How Well Do Constant-Maturity Treasuries Approximate the On-the-Run Term Structure?

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The par yield curve, or the relationship between yields to maturity and terms to maturity of securities trading at their par values, is a popular trading and pricing tool in the fixed-income market. This curve furnishes the on-the-run term structure of zero-coupon bond prices, the spot rate and forward rate curves, and the durations and convexities of various fixed-income securities.

The par yield curve is essential in many fixed-income applications. It is used to select the fixed rate on default-free interest rate swaps and to choose the coupons at which to issue new bonds at par (Sundaresan [1997]), and zero-coupon rates or prices are used to price liquid derivatives such as caps, floors, and interest rate swaps (Hull [1997]). In a monetary policy context, the par yield curve serves as an indicator of the market’s expectations regarding interest rates and future inflation rates. These rates are generally estimated from forward rates, making it easier to separate expectations for the short, medium and long term (Svensson [1994]).

In general, determining the par yield curve requires observations of either the yield on the most recently auctioned on-the-run Treasuries, or the yields on the constant-maturity series interpolated by the Department of the Treasury and published by the Federal Reserve Bank of New York in its H.15 release. Actual on-the-run yields are available in the market from issues with nine original maturities of between three months and 30 years. These securities trade close to par, and their yields are often taken as a proxy for par-bond yields.

Constant-maturity Treasury yields consist of the nine maturities plus a 20-year yield. These yields are read off a smooth curve fit through the nine on-the-run yields. The curve is constructed using a statistical model based primarily on cubic spline interpolation. Constant-maturity yields are not identical to market yields because of the smoothing process and the aging of on-the-run securities.

Bond researchers and practitioners often use constant-maturity yields as a surrogate for on-the-run yields. In the analysis of yield spreads (the spread between corporate bond yields and Treasury yields with a corresponding maturity), for example, researchers have used constant-maturity yields as a proxy for Treasury yields (see Das and Tufano [1996], Duffee [1998], and Duffie and Singleton [2000]). In the debt market and especially in the mortgage-backed securities market, constant-maturity yields are frequently used as indexes in variable-rate tranches such as floaters, inverse floaters, and notional IOs (see Fabozzi [1997]). Constant-maturity yields are also used in the pricing of liquid derivatives such as caps, floors, and interest rate swaps (Hull [1997]).

The purpose of this research is to examine the use of constant-maturity Treasuries as an alternative to actual yields observed in the Treasury market. This comparison is
based on pricing errors for in-sample and out-of-sample Treasuries. Pricing errors are determined by comparing market on-the-run prices with estimated prices from actual on-the-run yields and constant-maturity yields. To estimate the prices of on-the-run and constant-maturity securities, a term structure of spot rates is extracted from the observed yields. This can be done using bootstrapping. To implement bootstrapping, a function is fit through the on-the-run yields and the constant-maturity yields and spot rates are calculated recursively according to this function.

We use two functional forms to fit the yields: the cubic spline functional form based on the van Deventer and Imai [1996] specification, and the Nelson and Siegel [1987] functional form. Although we anticipate that the Nelson and Siegel model will price on-the-run Treasuries better than the cubic spline both in-sample and out-of-sample, we include it for comparative purposes, since it is one of the methods used by the Treasury Department to determine the constant-maturity yields.

I. TERM STRUCTURE ESTIMATION FROM ON-THE-RUN AND CONSTANT-MATURITY TREASURIES

On-the-Run Treasuries versus Constant-Maturity Treasuries

The input for term structure estimation methods consists entirely of yields obtained either from the on-the-run (OTR) Treasury yields or constant-maturity yields. These securities differ in that the reported OTR Treasury yields are market yields based on actual remaining time to maturity dates, while constant-maturity Treasury (CMT) yields are estimated yields that have been generated from a statistical model and correspond to the original on-the-run maturity dates.

On-the-run securities are the most recently issued and most liquid of the traded Treasuries. These securities are issues with nine original maturities of three and six months and one, two, three, five, seven, ten, and thirty years. Treasury securities with maturities of one year or shorter are issued as discount securities. These securities pay only a fixed amount at maturity and therefore sell for less than their par value. All other Treasury securities are sold as coupon securities. These securities pay interest every six months plus principal at maturity.

Discount securities are called Treasury bills, and they provide direct observation of spot rates. Treasury coupon securities are called Treasury notes or bonds, depending on the maturity of the issue. Coupon Treasuries issued with original maturities of between two and ten years are called Treasury notes; those with original maturities greater than ten years are called Treasury bonds. Treasury notes and bonds do not provide direct observation of spot rates.

Constant-maturity yields represent yields on Treasury securities at (fixed or constant) maturities of from three months to thirty years that are interpolated by the Department of the Treasury from the daily yield curve. This interpolation is based on the closing market bid yields of the actively traded Treasury securities in the over-the-counter market and calculated from the composites of quotations obtained by the Federal Reserve Bank of New York. Fixed or constant maturities, in this context, mean that this interpolation method provides a yield for a particular maturity even if no outstanding security has exactly that fixed maturity. Constant-maturity yields are not identical to market yields because of the smoothing process and the aging of OTR bonds.

Interpolating Functions Adapted for Estimating the Par Yield Curve

Cubic Spline Functional Form. Cubic spline interpolation is a simple approach based on the assumption that a cubic polynomial estimates the yield curve at each maturity gap. A spline can be thought of as a number of separate polynomials of \( y = f(x) \), where \( x \) is the range divided into segments joined smoothly at a number of knot points. Different segments have the same functional form with different parameters.

This interpolation method imposes a smooth joining of the different functions at the end of each points of the gap so that the entire yield curve is continuous with continuous first and second derivatives. For the purpose of this research, we use the van Deventer and Imai [1996] specification to fit a cubic spline to the par bond yield curve with knots placed at maturities identical to the original OTR Treasury maturities of from three months to thirty years.

To estimate the cubic spline functional form over any segment \( x \), we assume that, given bond yields \( y_0, y_1, y_2, \ldots, y_n \) consistent with maturities \( t_0, t_1, \ldots, t_n \), the yield on security \( i \) at time \( t \) can be expressed as a cubic polynomial such that
\[ y_i(t) = a_i + b_i t + c_i t^2 + d_i t^3 \]  

(1)

to the interval between \( t_i \) and \( t_{i-1} \). To estimate the cubic functional form at each knot point, we compute the coefficients \( a, b, c, \) and \( d \) in Equation (1) for all \( n \) intervals between the \( n + 1 \) data points. This gives us \( 4n \) unknown coefficients to estimate.

To solve for these coefficients over all knot points, we make use of the fact that these equations must fit the observable data points, and that the first and second derivative be equal at the \( n - 1 \) knot points. Because we lose two degrees of freedom by using the first and second derivatives, we obtain two end point constraints to complete the system. The first is chosen so that the yield curve is instantaneously straight at the left-hand side of the curve (i.e., \( y^{\prime\prime}(0) = 0 \)), and the second is chosen so that the yield curve is instantaneously straight at the longest maturity (i.e., \( y^{\prime\prime} = 0 \)).

Nelson and Siegel Functional Form. The Nelson and Siegel [1987] functional form was not developed for use as a bootstrapping method, but as a method for estimating a spot rate function from Treasury bills according to an assumed functional form for forward rates. Their spot rate functional form has been used for coupon yields, however (e.g., Barrett, Gosnell, and Heuson [1995]), and we use it in that manner.

Writing their spot rate equation as a coupon yield equation, we have

\[ y_T = \beta_0 + \beta_1 e^{-\frac{T}{\tau}} + \beta_2 \left( e^{-\frac{T}{\tau}} - 1 \right) \]  

(2)

where \( \beta_0, \beta_1, \beta_2, \) and \( \tau \) are the parameters to be estimated. The coefficients are estimated from a non-linear regression.

The three components in Equation (2) determine the appropriate choices of weights that can be used to generate yield curves of a variety of shapes. Interest rate-smoothing models of this type have the advantage of forcing the forward rate at the long end of the curve to a horizontal asymptote. They also avoid the problems inherent in spline-based models of choosing the “optimal” knot point specification, although these advantages are not without sacrifices. The trade-off is that these models, in theory, are less flexible than spline-based models and so may fit the data less well.

Bootstrapping

We use discrete-time and continuous-time bootstrapping to estimate the term structure of zero-coupon bond prices from on-the-run yields and constant-maturity yields. Bootstrapping refers to a recursive solution for spot rates at successive maturities, commonly six months apart.

The first step in implementing bootstrapping is to fit a curve (or a functional form) through the observed yields so that the yields can be computed for any time to maturity. This can be done by some interpolation technique. Two interpolation functional forms are used: the cubic spline functional form based on the van Deventer and Imai [1996] specification and the Nelson and Siegel [1987] functional form.

The next step in bootstrapping is to extract spot rates from the estimated yields by solving for these rates recursively. We use discrete-time bootstrapping based on cubic spline interpolation and continuous-time bootstrapping based on Nelson and Siegel interpolation to obtain spot rates and calculate zero-coupon bond prices at various maturities. Discrete-time bootstrapping is included for comparative purposes as it is a method used partially by the Treasury to estimate constant-maturity series from active OTR yields.

The “flat” price \( P_i(T) \) of a bond maturing in \( T \) periods, paying coupon \( C_i \), and of redemption value \( M_i \) is given by

\[ P_i(T) = C_i \sum_{t=1}^{T} \delta_t + M_i \delta_T \]  

(3)

where \( \delta_T \) is the present value of $1 payable in \( T \) periods. Equivalently, the bond price can written in terms of continuously compounded spot rates \( r_i \):

\[ P_i(T) = C_i \sum_{t=1}^{T} e^{-r_t t} + M_i e^{-r_T T} \]  

(4)

where

\[ r_t = -\frac{\ln(\delta_t)}{t} \]  

(5)

The continuously compounded instantaneous forward rate is given by the slope of the log discount function:

\[ F(t) = -\frac{d \ln(\delta_t)}{dt} \]  

(6)
The term structure of interest rates can be defined as the relationship between \( \delta_t \) and \( t \), or equivalently, the relationship between \( r_t \) and \( t \) or between \( F_t \) and \( t \). One relationship implies the other two.

Bootstrapping works as follows (see Hull [1997], Chapter 4). The first bond used in the recursion must have only one payment remaining, so that only one discount factor appears in Equation (3). Suppose this payment occurs in \( T_1 \) periods. Then the discount factor \( d_{T_1} \) is easily found from (3), since the other quantities are known (bond price, coupon, redemption value). The second bond in the recursion must have its first payment occurring in \( T_2 \) periods so that its value can be computed using \( d_{T_2} \). The second discount factor (call it \( d_{T_2} \)) is then found from (3), and so on.

Bootstrapping bond-by-bond in this manner limits the sample of bonds that can be used to those making payments at common dates. Estimating the term structure between these dates requires interpolation. In addition, deciding which bonds to include in the bootstrap is problematic, since bonds may differ in liquidity.

The so-called par yield methods of estimating the term structure seemingly avoid the limitations of bond-by-bond bootstrapping. The first step in a par yield method is to assume that the most recently issued, and therefore the most liquid, bonds sell at par, or \( P(T) = 100 \) for all \( T \). Note that the \( i \) subscript has been dropped because it is assumed that only one bond sells at par at each maturity.

Then the yield to maturity \( y_T \) of each par bond equals the coupon rate, or \( C = 100y_T \), and (3) can be written

\[
1 = y_T \sum_{t=1}^{T} \delta_t + \delta_T
\]  

(7)

The second step is to estimate a continuous function for the par yield curve, or the relationship between \( y_T \) and \( T \). The third step is to bootstrap along this hypothetical par yield curve to estimate \( \delta_T \) from Equation (7). In principle, there is now no limitation on the number of maturities for which \( \delta_T \) can be estimated. For example, it could be assumed that hypothetical par bonds pay coupons daily, and \( \delta_T \) could be calculated in daily increments. (Before doing so, the yield curve should be converted from the compounding convention in which it is estimated, such as semiannual, to daily compounding.) It is common, however, to bootstrap at three- or six-month intervals corresponding to the payment periods in most bond markets. In any case, we call this discrete-time bootstrapping.

Diamant [1993] has developed a continuous-time version of bootstrapping. Assuming that a theoretical par bond pays a continuous coupon, and using continuously compounded yields to maturity, the bond price equation can be written

\[
1 = \delta_T + y_T W_T
\]  

(8)

where

\[
W_T = \int_0^T \delta_t dt = e^{\int_0^T \delta_t dt} E_T
\]

\[
A_T = \int_0^T y_t dt
\]

\[
E_T = \int_0^T e^{\int_0^T \delta_t dt} dt
\]

From Equation (8), \( \delta_T \) is given by

\[
\delta_T = 1 - y_T e^{\int_0^T \delta_t dt} E_T
\]  

(9)

The bootstrapped discount function is obtained by solving these equations for successively larger values of \( T \) using numerical integration techniques.

II. DATA AND METHODOLOGY

Data Sources

The data used to estimate the term structure come from two sources: monthly on-the-run Treasury yields taken from the Center for Research in Security Prices (CRSP) bond files, and monthly constant-maturity yields taken from the Federal Reserve Bank statistics in its H.15 release. Yields are computed on the basis of Treasury bid prices using the continuous compounding convention.12

The study period is January 1, 1990, through December 31, 1997, sampled at the end of each month, making 96 samples in all. Treasury yields are taken from the nine on-the-run issues with original maturities of from three months to thirty years, for the period January 1, 1990, through December 31, 1993. Thereafter, the seven-year Treasury note was discontinued, so only eight of the original maturities are included. Constant-maturity yields are taken with comparable on-the-run original maturities.
Methodology

For interpolation in discrete time, we estimate the coefficients needed to compute the cubic spline functional form at every segment of the yield curve based on knot points that are identical to the original maturities of OTR yields. We establish two end point constraints: 1) the yield curve is instantaneously straight at the left-hand side of the yield curve; and 2) the yield curve is instantaneously straight at the longest maturity. For interpolation in continuous time, we estimate the non-linear regression parameters by regressing remaining time maturity against yield to maturity for the available on-the-run yields and fixed time to maturity against yield to maturity for constant-maturity yields.

The estimated parameters obtained from non-linear regression are then used as inputs to generate the yield curve for any time to maturity. A coupon-stripping procedure is used to extract and estimate the term structure of spot rates under both cubic spline and Nelson and Siegel [1987] functional forms.

Next, the estimated term structure of spot rates is used to price in-sample and out-of-sample securities. In-sample securities are the nine on-the-run Treasury issues with maturities of from three months to thirty years, with the exception of 1994 and thereafter where the seven-year Treasury note was taken out of circulation and only eight maturities are used. These securities are used to obtain the functional form that allows the yield to maturity to be computed for any time to maturity.

Out-of-sample securities consist of five securities from each sample date. These are the first off-the-run securities (i.e., the securities closest in terms of maturity to the on-the-run securities) with maturities of approximately two, three, five, ten, and thirty years. Out-of-sample securities allow us to determine how well our models approximate the on-the-run term structure.

We compare the results of the cubic spline model in terms of pricing accuracy both in-sample and out-of-sample with the Nelson and Siegel model using both on-the-run Treasuries and constant-maturity Treasuries. This will allow us to examine pricing errors according to both models and for both data sets.

In addition, we examine a 30-year bond that has been outstanding for approximately 15 years; we call this the off-off-the-run security. The idea is to find how well the on-the-run and constant-maturity yield curves determine the prices of the off-off-the-run security. That is, how well do the data sets and methods price an out-of-sample security that has been outstanding for an extended period of time whose price may reflect low liquidity? While it is expected that the 15-year bond would produce more error in term structure estimation, this exercise is useful in determining the magnitude of error and therefore the effect of illiquidity on bond prices.

Descriptive Statistics

Exhibit 1 shows descriptive statistics for differences between on-the-run yields and constant-maturity yields, in basis points, for the period January 1, 1990, through December 31, 1997. Included are the mean, median, standard deviation, maximum, and minimum values. Yield differences are computed based on semiannual compounding.
that, on average, constant-maturity Treasuries are quoted on the basis of yields that are higher than on-the-run yields. The discrepancy in yields is primarily the result of maturity differences between the securities used in on-the-run and constant-maturity data sets.

For individual maturities, the results vary. At the maturities of three and six months and thirty years, the mean differences in yields are between 0.5 to 2 basis points, with a standard deviation of about 3 to 4 basis points. Individual maturities of one, two, three, and seven years have mean differences in yields of about 4.0 to 6.5 basis points, an increase in yield differences on average of about four times the other maturities. The greatest fluctuations in yield differences denoted by maximum and minimum values in Exhibit 1 occur at the intermediate range of three years and seven years, and at the long end of the yield curve at 30 years.

One possible reason is that differences between on-the-run yields and constant-maturity yields at various maturities may arise as a result of the differences in the frequency of Treasury auctions. For example, Treasury securities of maturities of three months and six months are auctioned on a weekly basis, while maturities of three years and thirty years are auctioned quarterly and semi-annually, respectively. These differences can cause interest rates to fluctuate more widely, causing yield differences between OTR and constant-maturity series to widen.

Finally, one interesting result is that at the long end of the yield curve the sign of the mean difference in yields is positive, indicating that yields obtained from the on-the-run data are higher, on average, than constant-maturity yields. This may be the result of the smoothing process that the Treasury uses to compute yields at that maturity.

Next, we are interested in reducing yield discrepancies, for a given maturity, between on-the-run Treasuries and constant-maturity Treasuries in order to reduce price estimation errors. Exhibit 2 shows descriptive statistics for differences between on-the-run yields and interpolated constant-maturity yields, in basis points, for the period January 1, 1990, through December 31, 1997. We fit constant-maturity yields using the Nelson and Siegel [1987] functional form and compute the yields at on-the-run maturities.

The results suggest that yield differences between on-the-run and constant-maturity Treasuries can dramatically decrease when we use an interpolation function such as Nelson and Siegel. Over all maturities, the differences in yields have a mean of 0.25 basis points and a standard deviation of 0.07 basis point, with maximum and minimum values of about 13 basis points. This represents a significant reduction in yields between the on-the-run yields and constant-maturity yields from the results in Exhibit 1 (reduction in mean yields of about eight times, in standard deviation of mean yields of about 64 times, and in maximum and minimum values of mean yields of about 1.5 times).

For individual maturities, the same is true. Significant reductions in yield differences are achieved in all maturity segments of the yield curve. This suggests that yield differences are attributable primarily to maturity differences that can be reduced by using a method such as the continuous-time bootstrapping with the Nelson and Siegel as the interpolation function.

### III. EMPIRICAL RESULTS

#### In-Sample Performance Evaluation

Exhibit 3 summarizes in-sample coupon bond price estimation results for the period January 1, 1990, through December 31, 1997. The results are estimated on
the basis of discrete-time and continuous-time bootstrapping using cubic spline and Nelson and Siegel [1987] functional forms for both on-the-run yields and constant-maturity yields.

For pricing accuracy, we report weighted mean percentage error (WMPE), weighted standard deviation of percentage error (WSPE), and weighted mean absolute error (WMAE) as the forecast statistics. The WMPE statistic measures unbiasedness, while the WSPE and WMAE statistic measure magnitude of error. We follow Coleman, Fisher, and Ibbotson [1995], and use the inverse of the square root of duration as the weighting scheme.

The rationale for reporting weighted (or scaled) errors is that our estimated errors are heteroscedastic in prices. As a result, price errors must be scaled to deflate the influence of the less precise observation (in this case where the data gaps are wide, such as between ten and thirty years).

The results show that using the Nelson and Siegel model with input from on-the-run yields produces error statistics that are routinely smaller than those produced by constant-maturity yields. For example, the WMAE over all maturities using on-the-run yields is about 0.06%. This translates into an error of $0.06 per $100 face value. The highest WMAE for all maturities for the constant-maturity yields is about 0.12%, which translates into an error of $0.12 per $100 face value. This is an increase in error of about two times for constant-maturity yields over OTR yields.

In terms of maturity ranges, it appears that when there are concentrated data points (such as in the range of zero to two years), the difference in pricing errors between the models is small ($0.004 per $100 face value). When the gap is great, between ten years to thirty years, the differences in pricing are large ($0.23 per $100 face value).

In the intermediate ranges of two to five years, the differences vary. For example, in the two- to five-year maturity range, the difference in magnitude of error is about $0.04 per $100 face value, as opposed to a magnitude of error of $0.09 per $100 face value in the five-to ten-year maturity range. The discrepancy in pricing errors may be the result of maturity differences and the aging of the on-the-run bonds as auctions of different types of OTR securities vary in terms of frequency.

In terms of unbiasedness, on-the-run yields produce less bias than constant-maturity yields, based on the Nelson and Siegel functional form, over the overall maturity range. In terms of maturity ranges, the bias is slightly greater for on-the-run yields at the short and intermediate ranges, and significantly less at the long end (ten to thirty years). For example, in the short maturity range of less than two years and two to five years, the WMPE for the on-the-run yields is about −0.04% and −0.03%, respectively. This translates into errors of about $0.04 and

### Exhibit 3

<table>
<thead>
<tr>
<th>In-Sample Coupon Bond Price Estimation Results</th>
<th>January 1990 – December 1997</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant-Maturity Series</td>
<td>On-The-Run Series</td>
</tr>
<tr>
<td>Cubic Spline Nelson and Siegel</td>
<td>Cubic Spline Nelson and Siegel</td>
</tr>
<tr>
<td>All Maturities</td>
<td></td>
</tr>
<tr>
<td>WMPE</td>
<td>WMPE</td>
</tr>
<tr>
<td>-0.02755^</td>
<td>-0.05670</td>
</tr>
<tr>
<td>0.16651†</td>
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</tr>
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<tr>
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<td>0.24666†</td>
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</tr>
</tbody>
</table>

^ Significantly different from Nelson and Siegel central tendency at 5% level.
† Significantly different from Nelson and Siegel dispersion at 5% level.

Weighted mean percentage error (WMPE), weighted standard deviation of percentage error (WSPE), and weighted mean absolute error (WMAE), with the weight the inverse of square root of duration. Tests for statistical significance are the Wilcoxon rank sum test for central tendency and the Siegel-Tukey test for dispersion. Data taken from daily CRSP bond file for OTR yields and from the Federal Reserve Bank of New York for constant-maturity yields. Par value = $100.
better approximate the on-the-run term structure than models that use constant-maturity yields, both in terms of magnitude of error and unbiasedness. The Nelson and Siegel functional form also provides fewer pricing errors than the cubic spline functional form over all maturity ranges and in almost all cases.

Further, when we examine the estimated price errors for each sample, we find that the errors for each method are non-normal (according to the Jarque-Bera [1987] test), which is perhaps to be expected due to the interpolation error. As a result, we test for differences in central tendency (bias) and dispersion using the non-parametric Wilcoxon rank sum and Siegel-Tukey tests.

The results indicate that errors produced by the cubic spline model are statistically significant with respect to central tendency and dispersion compared to the Nelson and Siegel model over all maturity ranges and in all sub-maturity averages.

### Out-of-Sample Performance Evaluation

Exhibit 4 summarizes out-of-sample coupon bond price estimation errors for the sample period January 1, 1990, through December 31, 1997. We look at the models’ abilities to predict bond prices out-of-sample. For comparative purpose, we use the Nelson and Siegel functional form only because it continues to produce errors that are smaller than and statistically different from the cubic spline model in all maturity ranges.

The out-of-sample results are strikingly similar to the in-sample results, but with a greater forecast error. Over all maturities, the out-of-sample magnitude of error increases over the in-sample error by about two to three times. The out-of-sample magnitude of error (WMAE) for the Nelson and Siegel model, using on-the-run yields as input, over all maturities is about $0.05 per $100 face value less than the same model with constant-maturity yields as input.
In terms of unbiasedness, the Nelson and Siegel model with on-the-run yields produces a negative bias out of sample, $0.07 per $100 face value, about 1.9 times the in-sample error. In terms of maturity ranges, the Nelson and Siegel model with on-the-run securities as input produces better pricing (less magnitude of error) than the same model with constant-maturity series as input. This is true for all five maturities.

Finally, for the off-off-the-run bond, the results are similar. The Nelson and Siegel model with on-the-run yield prices the off-off-the-run bond better in terms of magnitude of error and unbiasedness than the same model with constant-maturity yields. On an absolute basis, the magnitude of error and unbiasedness for the Nelson and Siegel with on-the-run yields, is about $0.17 and $0.28 per $100 face value compared to $0.31 and $0.39 per $100 face value for the same model with constant-maturity yields.

Overall, using the Nelson and Siegel model with on-the-run yields as input produces less error over all maturity ranges than the same model with constant-maturity yields.

Implications

The tests on in-sample, and out-of-sample data indicate that models that consider on-the-run Treasury yields better approximate the term structure of interest rates in terms of pricing than models that use constant-maturity Treasury yields. While the differences between OTR yields and constant-maturity yields are significant, they can be improved by using a different methodology to estimate the Treasury par yield curve: a continuous bootstrapping technique. It can be easily implemented, and it is robust and consistent with market data.16

IV. SUMMARY AND CONCLUSION

We have compared the use of constant-maturity Treasuries as an alternative to on-the-run yields obtained from the Treasury market, looking at pricing errors in the period January 1, 1990, through December 31, 1997. For pricing, we use bootstrapping based on discrete time and continuous time with two functional forms: the cubic spline functional form based on the van Deventer and Imai [1996] specification and the Nelson and Siegel [1987] functional form. We include bootstrapping based on the cubic spline functional form for comparative purposes, because it is one method the Treasury uses in its estimation of constant-maturity series.

The results show that on-the-run yields rather than constant-maturity yields better approximate the term structure both in-sample and out-of-sample according to the Nelson and Siegel specification. For in-sample data, over all maturities, the error bias (WMPE) is about $-0.04\%$ and $-0.06\%$ with magnitudes of error (WMAE) of $0.06\%$ and $0.12\%$ for OTR yields and constant-maturity yields, respectively. For out-of-sample securities, the results are similar, only uniformly higher. Over all maturities, the WMPE is about $-0.07\%$ and $-0.11\%$ with standard deviations of $0.16\%$ and $0.20\%$ for OTR yields and constant-maturity yields, respectively. The results also hold within maturity ranges both in-sample and out-of-sample. The Nelson and Siegel errors are statistically significantly different from the cubic spline errors for both OTR and constant-maturity yields at the 5% level.

We also investigate the usefulness of on-the-run yields versus constant-maturity yields for pricing off-off-the-run securities. The idea is to find how well the on-the-run term structure prices securities with different degrees of liquidity. The results are similar, and suggest that, for less liquid bonds, constant-maturity yields produce about 1.4 times the magnitude of error over the study period than on-the-run yields. One should be cautious about using constant-maturity yields as an alternative to OTR yields.

Our main finding is that the Treasury Department can dramatically improve its estimation of constant-maturity series by adopting a different methodology from the one it now uses. According to our results, a continuous bootstrapping methodology with the Nelson and Siegel functional form as the interpolation method can reduce pricing errors dramatically. This certainly has implications for both academicians and practitioners who widely use constant-maturity Treasuries as a surrogate for on-the-run Treasuries.

ENDNOTES

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1For example, the Bank of England uses the Mastronikola [1991] model for monetary policy purposes, which is a par yield curve model that assumes a natural cubic spline as a functional form to fit par bond yields.

2Three and six months, and one, two, three, five, seven, ten, and thirty years. In October 1993, the Treasury discontinued issue of seven-year Treasury notes. Therefore, any data after October 1993 should include only eight on-the-run Treasuries.
In general, these securities do not trade exactly at par, as interest rates have changed since their original issuance. Longer-term securities, which are issued more infrequently, may trade substantially away from par.

The main issue here is availability. Constant-maturity yields can be easily obtained from any of the regional Federal Reserve Bank web sites.

Because spot rates are directly observable for maturities of one year or shorter, corresponding to the maturities of Treasury bills, the statistical approach to term structure estimation is largely the problem of extracting discount factors, spot rates, and forward rates from coupon bond data.

For example, in estimating the 20-year constant-maturity, the Treasury incorporates the prevailing market yield on an outstanding Treasury bond with approximately 20 years to maturity.

At the close of the trading day, composite closing market bids (yields) on outstanding Treasury securities are reported by five U.S. government securities dealers to the Federal Reserve Bank of New York and plotted by the Treasury on a graph. The horizontal axis shows the maturity date of each reported security, and the vertical axis measures the yield; a continuous curve is fitted either using a statistical model or by hand through the plotted points.

The word smooth, in the case of a piecewise n-degree spline, is generally taken to mean that the \((n - 1)\)-th derivatives of the functions on either side of each knot point are continuous.

For more information, see van Deventer and Imai [1996, pp. 45-47].

Jordan and Mansi [1999] find that the Nelson and Siegel functional form performs better than other yield-smoothing methods.

Jordan and Mansi [1999] show that the continuous bootstrapping methods price OTR securities better than the discrete bootstrapping methods both in-sample and out-of-sample.

In the CRSP data set, yields are calculated according to the average of bid and ask prices of Treasury securities. To make yields comparable, OTR Treasury yields are recalculated on the basis of bid prices only.

The rationale for using the first off-the-run securities is that these securities are closest to the on-the-run Treasuries in terms of liquidity.

The maturity of the off-off-the-run security lies in the data gap range of the on-the-run methods.

This exercise is also useful because the Treasury publishes a constant-maturity 20-year rate that could entail an illiquidity factor, so its estimated price based on the on-the-run Treasury yield curve may diverge considerably from its actual price.

Various central banks are adopting new techniques to improve their estimation of the yield curve. The Bundesbank, for example, has adopted a modified version of the Nelson and Siegel [1987] model, moving away from its in-house model, for monetary policy purposes and for estimation of the par yield curve. For more information on this topic, see “Estimating the Term Structure of Interest Rates” [1997].

REFERENCES


Notes: Coefficients are above; t-values are in parentheses. The dependent variable is an indicator variable that takes on the value of one in the month of a move. Loans that default or refinance are included in the sample (with indicator value = 0) until their observation period is truncated. The variables in the model are the ratio of the seven year Treasury rate at origination to the seven-year Treasury rate in the month of observation (C/R); the tenure in the house (age and age$^2$ and the indicator that tenure in the house is longer than the time since origination); the mortgage credit score, the loan-to-value ratio, the unexplained difference between the rate on a given loan and the market rate, an indicator for ARM loans and third party origination.