DYNAMIC ASSET ALLOCATION
FOR STOCKS, BONDS AND CASH

Isabelle Bajeux-Besnainou*, James V. Jordan** and Roland Portait***

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*Department of Finance
School of Business and Public Management
George Washington University
2023 G ST, NW
Washington DC 20052
bajeux@gwu.edu
202-994-2559 (tel) 5014 (fax)

**National Economic Research Associates
1255 23rd St. NW
Washington DC
james.jordan@nera.com
202-466-9263 (tel) 9211 (fax)

***CNAM and ESSEC
Chair of Finance; CNAM
2 rue Conté, Paris 75003 France
rolcot@wanadoo.fr

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Abstract

Closed-form solutions for HARA optimal portfolios are obtained in a dynamic portfolio optimization model in three assets (stocks, bonds and cash) with stochastic interest rates. A Vasicek-type model of stochastic interest rates with a correlated stock price is assumed. The HARA solution can be expressed as a buy and hold combination of a zero-coupon bond with maturity matching the investor’s horizon and a “CRRA mutual fund,” which is the optimal portfolio for a CRRA investor expressed in terms of the weights on cash, stock, a constant-duration bond fund and the (redundant) bond with maturity matching the investor’s horizon (a generalization of the Merton’s (1971) result of constant weight in stock for a CRRA investor, derived for two assets with constant interest rates). This simple characterization facilitates insights about investor behavior over time and under different economic scenarios and allows fast computation time (without simulation or other numerical methods). We use the model to provide explanations of the Canner-Mankiw-Weil (1997) asset allocation puzzle, the use of “convex” (momentum) and “concave” (contrarian) investment strategies and other features of popular investment advice. The model illuminates clearly the role of the different market parameters, the composition of initial investor wealth, and relative risk aversion in portfolio strategies.
I. Introduction

Merton’s (1971) breakthrough in continuous-time portfolio theory allowed the analysis and understanding of dynamic hedging (through dynamic fund separation theorems) when different risk factors are considered in the economy. Further research in this framework has led to a number of conclusions about the nature of optimal dynamic portfolios in particular settings defined by investor preferences and the components of the investment opportunity set. For example, it is now well known from Samuelson (1969) and Merton (1971) that for investors displaying constant relative risk aversion (hereafter CRRA), asset prices following a geometric brownian motion and a constant interest rate, the optimal strategy is to maintain constant weights over time. Samuelson obtained decreasing equity weights for CRRA preferences by assuming mean-reverting stock returns (see Samuelson (1991)) or with a minimum wealth constraint for geometric brownian motion (Samuelson (1989)), in both cases with a constant interest rate. Bodie, Merton and Samuelson (1992) demonstrated the optimality of decreasing equity weights through time by introducing human capital and variable work effort into the Merton (1971) consumption-investment model, again with a constant interest rate. Until recently, there were few results for the case of stochastic interest rates and virtually no closed-form solutions.

Bajeux-Besnainou and Portait (1998) derived closed-form solutions for three-asset, dynamic mean-variance allocations in a Vasicek-type market in which the single state variable is the short-term interest rate, the stock index follows a state-dependent process correlated with interest rates, and the bond alternative is either or both a zero-coupon bond maturing at the horizon and a constant-maturity zero-coupon bond fund.¹ The same model has been used in Bajeux-Besnainou, Jordan and Portait (2001) for a numerical illustration of the theoretical
explanation of the Canner, Mankiw and Weil (1998) puzzle. Sorensen (1999) considers a Vasicek bond market with three assets (cash, stocks and a bond with maturity matching the investor’s horizon) and a CRRA investor. The paper provides closed form expressions for the optimal weights and contains an analysis of portfolio policies involving transactions at discrete points of time (by an immunization procedure and by quasi-dynamic programming); it contains an application to bond portfolio selection based on deviations between theoretical and actual bond prices. Deelstra, Grasselli and Koehl (2000) consider the market framework of Bajeux-Besnainou and Portait (1998) (involving cash, stocks and a bond with maturity matching the investor’s horizon or a bond fund with constant duration), but with a Cox-Ingersoll-Ross (CIR) bond market. Closed form solutions for the CRRA optimal strategies are derived. Liu (1999) starts from a very general framework (multi-factors model, general utility function with intermediate consumption, incomplete markets) and characterizes portfolio and consumption solutions by PDE’s a la Merton; specializes progressively the framework (such as CRRA investors), and provides examples for some specific diffusion processes including the Vasicek and CIR models. These models are applied to bond portfolio selection.

Bielecki, Pliska, and Sherris (1998) model security price dynamics as a multi-dimensional diffusion process with constant volatilities (as in Langetieg (1980)). The risk premium is a linear combination of n factors with the factors following a multi-dimensional Ornstein-Uhlenbeck process. They investigated the case of two uncorrelated Brownian motions governing the stock index and the risk premium. The portfolio was optimized over an infinite horizon with a “risk sensitive criterion”; closed form solutions were given for the initial
allocations. Omberg (2001) solved the CRRA case also in a multi-factor framework a la Langetieg (1980), which allows in particular the analysis of inflation risk. Inflation risk has also been central to the analysis conducted in Campbell and Viceira (2001), who again are using a Langetieg (1980) type of model for the stochastic interest rates but a recursive utility function a la Epstein and Zin (1989, 1991). Moreover, they show that an infinitely risk averse investor will hold only a real consol bond. Wachter (1999) generalizes this latter result to any HARA utility function and any investment horizon but again in the case of an infinitely risk averse investor with intermediate utility over consumption.

Recently, a number of authors have also tackled the question of stochastic interest rates from an numerical or empirical perspective. In particular, Barberis (1999) has used numerical methods in the case of asset return predictability and Brandt (1999) has developed a nonparametric approach to study optimal portfolio and consumption choices in the case of a CRRA utility function when these decisions depend on variables that forecast time varying investment opportunities.

In this paper, we concentrate on the simplest case of stochastic interest rates, a one-factor Vasicek-type model, and we derive closed-form solutions in a dynamic optimization model for investors displaying hyperbolic absolute risk aversion (HARA). The investment set includes, four assets: cash, stock, one bond fund (represented by a zero coupon bond with constant maturity) and a zero coupon bond with maturity matching the investor’s horizon T (the risk free asset). These assumptions imply dynamically complete markets (although one asset is redundant). This investment set, previously considered in Bajeux-Besnainou and Portait (1998),
Deelstra, Grasselli and Koehl (2000), and Bajeux-Besnainou, Jordan and Portait (2001), presents several advantages:

- It avoids the undesirable result obtained when only a bond matching the investors horizon is considered: the optimal weight on this bond tends to infinity and the optimal weight on cash tends to minus infinity as time gets closer to the horizon T. This problematic (though explainable) result makes comparisons with actual portfolio advice almost impossible. Moreover, the solution (optimal weights) when only one bond is considered involves the inverse of a time varying variance covariance matrix and is not directly interpretable. In contrast, the model involving the two bonds yields expressions for the optimal investment strategies which are extremely simple (constant weights in the CRRA case), hence easy to interpret.

- It allows the optimal solutions to be compared to the allocations of the popular advice. Indeed, the popular advice on strategic asset allocation always involves a bond fund, but not a bond matching the maturity of the investor (probably because the latter is investor-specific and the popular advice is intended for a wide spectrum of investors). A model involving assets which are different from those actually considered in the advice of practitioners would be therefore inadequate to assess the realism of the theoretical results or the theoretical validity of the popular advice.

Another difference between our model and most of the previous related work is that HARA utility is assumed (instead of the more restrictive CRRA) with the purpose of studying the effects of wealth with varying relative risk aversion: for instance, the way relative risk aversion varies
with wealth is the key explanation of the investor’s choice between a contrarian and a momentum strategy, as we show in the paper.

With the solutions, we are able to investigate the implications of different preference assumptions and market parameters, as well as the rationality of certain aspects of the popular advice about asset allocation for the three asset classes stocks, bonds, and cash. We start in the case of CRRA utility and prove that the optimal dynamic strategy can be expressed as a constant weight combination of cash, a stock fund, a bond fund with constant duration and a zero-coupon bond with a maturity matching the investor’s horizon. We call this the “CRRA mutual fund;” it is rebalanced continuously to maintain constant weights in the four assets. Alternatively, when only three assets are considered (dropping the zero-coupon bond maturing at the horizon), the investor maintains a constant allocation in stocks, while the bond fund allocation is shifted deterministically to cash over time. This is a generalization of the Merton (1971) model (geometric brownian stock prices and constant interest rates) from two assets to three assets with stochastic interest rates.

For general HARA preferences the optimal strategies can be characterized as a buy and hold combination of the zero coupon bond and the CRRA mutual fund. This simple characterization of the optimal strategies, in a context of stochastic interest rates, allows fast computation time, insights into the characteristics of optimal portfolios, easy interpretations of the portfolio strategy paths and of the investor’s behavior in different economic scenarios and a clear elucidation of the role of risk aversion in portfolio strategies. The bond-stock allocation puzzle of Canner, Mankiw and Weil (1997) (that the bond to stock weight ratio increases with risk aversion in the popular investment advice, in apparent contradiction with standard two fund
separation) is easily solved. The key role of risk aversion, in combination with the proportion of the considered portfolio in the investor’s total wealth, in explaining the convexity of an optimal strategy (more risky assets are purchased after prices increase) is also demonstrated.

The paper is organized as follows. The framework, notation and methodology are specified in Section II. The HARA results are provided in Section III. The effects of the investor’s preferences, market parameters and market evolution on optimal strategies are studied in section IV. Section V is a conclusion.

II. Framework, notation and methodology

Markets are assumed arbitrage-free, frictionless, continuously open between 0 and $T$, and dynamically complete. The stochastic structure and the information flow is represented, as usual, by the probability space $(\Omega, F, P)$ where the filtration $F=(F_t), t \in [0, T]$ satisfies the usual conditions (see, for example, Duffie (1992)). The measure $P$ is called the historical, or true, probability.

The investment set consists of four assets (one is redundant in this model): an instantaneously riskless money market fund (“cash”) with price at $t$ given by $M(t)$; a stock index fund with price $S(t)$; a $1$ par value constant-maturity, zero-coupon bond fund maturing at $t+K$ with price $B_K(t)$; and a $1$ par value zero-coupon bond maturing at date $T$ with maturity $T-t$ and price $B_{T-t}(t)$. The bond fund is continuously rebalanced to maintain a constant maturity of $K$ throughout the period 0 to $T$. The $(T-t)$-maturity bond is the risk-free asset for an investor with a horizon of $T$; its maturity continually decreases as time passes. Since the term structure is driven by only one state variable, the bonds are perfectly correlated and therefore redundant.
The notation \( M, S, B_K \) and \( B_{T-t} \) is also used to represent the four assets when there will be no confusion between this use of the notation and the prices of the assets.

The investment opportunity set depends on one state variable, the instantaneous riskless interest rate \( r \) which follows an Ornstein-Uhlenbeck process with constant parameters given by

\[
dr(t) = a_r (b_r - r(t))dt - \sigma_r dz_r
\]

where \( a_r, b_r \) and \( \sigma_r \) are positive constants and \( z_r(t) \) is a standard brownian motion.\(^3\) The market price of interest rate risk \( \lambda_r \) is assumed constant; since markets are assumed arbitrage free this yields a Vasicek-type bond market.

The stochastic processes for the securities are given by

\[
\frac{dM(t)}{M(t)} = r(t)dt
\]

\[
\frac{dS(t)}{S(t)} = \left( r(t) + \Theta_S \right) dt + \sigma_S dz + \sigma_z dz_r
\]

\[
\frac{dB_K(t)}{B_K(t)} = \left( r(t) + \Theta_K \right) dt + \sigma_K dz_r
\]

\[
\frac{dB_{T-t}(t)}{B_{T-t}(t)} = \left( r(t) + \Theta_{T-t}(t) \right) dt + \sigma_{T-t}(t) dz_r
\]

where \( \Theta_S \) and \( \Theta_K \) are the risk premia of the stock fund and the bond fund, respectively; \( \sigma_S, \sigma_2 \) and \( \sigma_K \) are positive constants; \( dz \) is a standard brownian motion orthogonal to \( dz_r; \Theta_{T-t}(t) \) is the time-dependent risk premium of the bond maturing at \( T \); and \( \sigma_{T-t}(t) \) is the time-dependent
volatility of this bond. Also, and without loss of generality, the initial prices of the money market fund and the stock index fund are normalized to unity.

The volatility and risk premium of the bond fund are given by Vasicek's formulae

$$\sigma_K = \sigma_r \left(1 - e^{-\sigma_r K} \right)$$  \hspace{1cm} (6)

and

$$\theta_K = \sigma_K \lambda_r$$  \hspace{1cm} (7)

The volatility and risk premium of the bond maturing at $T$ are also given by (6) and (7) if $\sigma_T(t)$ is substituted for $\sigma_K$ and $T-t$ is substituted for $K$; $\sigma_K$ and $\theta_K$ are constant; $\theta_T(t)$ and $\sigma_T(t)$ are deterministic functions of time. The market price of stock market risk (\lambda) is assumed constant, hence the risk premium of the stock index fund, $\theta_S$, is also constant and given by

$$\theta_S = \sigma_r \lambda + \sigma_z \lambda_r$$  \hspace{1cm} (8)

The variance-covariance matrix of stock fund and bond fund returns is then given by

$$\Gamma = \begin{pmatrix} \sigma_1^2 + \sigma_z^2 & \sigma_z \sigma_K \\ \sigma_z \sigma_K & \sigma_K^2 \end{pmatrix}$$  \hspace{1cm} (9)

The variance-covariance matrix and its inverse\footnote{\textsuperscript{5}} can be written in terms of the bond maturing at $T$ with the appropriate substitutions (changing $K$ into $T-t$) for the bond's time-varying risk premium and volatility. This setting is a simple extension of the Black-Scholes framework incorporating stochastic interest rates correlated with stock returns.

Portfolio strategies can be defined by their weights on the four assets denoted $x_i(t)$, $i = M, S, K, \text{and } T-t$. Initially in this paper, strategies are expressed in terms of a three-asset
portfolio excluding the bond maturing at $T$, defined by the vector $\chi'(t) = (x_S(t), x_K(t))$.

Because the portfolio weights sum to one, the weight in cash is implicit and given by $x_M(t) = 1 - x_S(t) - x_K(t)$. Later, four-asset portfolio descriptions are helpful.

For economic and mathematical reasons, some admissibility conditions must be imposed on dynamic portfolio strategies. Among these admissibility requirements the weight vector $\chi(t)$ must satisfy integrability conditions and the terminal payoff (the return when initial prices are normalized to one) must display finite mean and variance. Only self-financing strategies are considered in this paper; the self-financing condition can be written

$$\frac{dX(t)}{X(t)} = x_M(t) \frac{dM(t)}{M(t)} + x_S(t) \frac{dS(t)}{S(t)} + x_K(t) \frac{dB_K(t)}{B_K(t)}$$

where $X(t)$ is the portfolio value at $t$. Among the admissible self-financing strategies, one is of special interest: the numeraire portfolio (Long(1990)), also known as the logarithmic or growth-optimal portfolio (Merton (1992)) since it maximizes expected logarithmic utility. The weights $h$ of the logarithmic portfolio are given by $h = \Gamma^{-1} \theta$.

In the three-asset framework (choosing the constant-maturity, $K$-maturity bond fund in lieu of the $(T-t)$-maturity bond), $h' = (h_S, h_K)$ is given by

$$h = \Gamma^{-1} \theta = \frac{1}{\sigma_S^2 \sigma_K^2} \begin{pmatrix} \sigma_K^2 & -\sigma_S \sigma_K \\ -\sigma_S \sigma_K & \sigma_S^2 + \sigma_K^2 \end{pmatrix} \begin{pmatrix} \theta_S \\ \theta_K \end{pmatrix}$$

(10)

An advantage of expressing the portfolio in terms of the constant-maturity bond fund is that these weights are constant over time. Using (6) and (7), the weights can be written
Setting its initial value $H(0)$ to unity, the price $H(t)$ of the growth optimal portfolio is then given by

$$H(t) = \exp\left[\int_0^t r(s)ds + \frac{1}{2}\left(\lambda^2 + \lambda, r^2 \right)t\right] \exp\left[\lambda(z(t) - z(0)) + \lambda, (z_r(t) - z_r(0))\right]$$

The logarithmic portfolio has the remarkable property that all prices of self-financing portfolios or securities, relative to $H(t)$, are martingales with respect to the historical (true) probability $P$ (e.g., $E[X(T)/H(T)] = X(0)$, where the expectation is taken at date 0); see, for example, Merton (1992) and Long (1990)). This property implies that a portfolio paying $1/H(T)$ can be interpreted as the “pricing kernel.”

Another property of the logarithmic portfolio $h$ is that it is one of the funds involved in the composition of any optimal portfolio. Indeed, we know from Merton’s fund separation theorems (1971, 1973, 1992) that, in a one state variable framework, the weights $\lambda^*$ on risky assets $S$ and $B_K$ of any optimal portfolio $(\lambda^*, X^*)$ can be written

$$\lambda^* = \gamma_1(X^*(t), r(t), t)h + \gamma_2(X^*(t), r(t), t)b_K$$

where $\gamma_1$ is the relative risk tolerance, $\gamma_2$ is also determined by the utility function and $b_K$ is the vector $(0, 1)$ (the weights on $M$ and $B_K$, respectively).

It is clarifying at this point to keep track of the allocations to all four assets. The notation $D[x_M(t), x_S(t), x_K(t), x_T(t)]$ defines a dynamic portfolio strategy by specifying the four allocations in securities $M$, $S$, $B_K$, and $B_T$, at date $t$, where the four weights sum to one. It should be noted that there is no unique specification of strategy $D$ because of the redundancy of
securities $B_K$ and $B_{T-t}$. For instance, $\$1$ invested in $B_{T-t}$, denoted $D[0, 0, 0, 1]$, is equivalent to $\$\sigma_{T-t}(t)/\sigma_K$ invested in $B_K$ and the complement $\$(1- \sigma_{T-t}(t)/\sigma_K)$ in $M$, or $D[1-\sigma_{T-t}(t)/\sigma_K, 0, \sigma_{T-t}(t)/\sigma_K, 0]$. In this notation, the weights of the logarithmic portfolio strategy can be written $D[1-h_S-h_K, h_S, h_K, 0]$ which is equivalent to the previously used more compact notation $(h_S, h_K)$.

To obtain closed form solutions for the optimal portfolios, we use the transformation of the dynamic problem into a static problem as developed in Pliska (1982, 1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989). Following this approach, we first obtain the equation of the optimal portfolio value and then the optimal strategy (weights) yielding the optimal portfolio value is derived.

The first step of the optimization procedure for an investor displaying a utility function $U$ on his terminal wealth $X(T)$ is thus to solve the optimization program $(P)$:

$$
(P) \quad \max_{X(T)} E[U(X(T))] \quad \text{s.t.} \quad E\left(\frac{X(T)}{H(T)}\right) = X_0
$$

The budget constraint expresses the fact that, at time 0, the investor “buys” a contingent claim, yielding $X(T)$, whose price is equal to her initial endowment of $X_0$. The solution is the optimal terminal wealth $X^*(T)$ which satisfies the first order conditions that write:

$$
U'(X^*(T)) = \kappa/H(T)
$$

(13)

where $\kappa$ is the lagrangian multiplier of the budget constraint. Whenever $U$ is specified, (13) yields $X^*(T)$. The second step (determining the strategy (weights) yielding $X^*(T)$) is a more complicated task to which we devote most of the next section.
III. Optimal strategies for HARA investors.

We begin by considering a CRRA utility function before solving the general HARA problem. The utility of a CRRA investor can be written \( \frac{X(T)^{1-\gamma}}{1-\gamma} \), where \( \gamma \) is the relative risk aversion. Program \((P)\) yields the following optimal terminal wealth \( X^*(T) \) through the first order conditions (13) and the budget constraint:

\[
X^*(T) = kH(T)^{1/\gamma} \tag{14}
\]

where \( k = \frac{X_0}{E[H(T)^{1/\gamma-1}]} \).

In a second step the task is to determine the optimal strategy (weights) attaining \( X^*(T) \).

This strategy is described in Proposition 1.

**Proposition 1**

- The optimal strategy of the CRRA investor (CRRA(\(\gamma,T\)) mutual fund), written in terms of the logarithmic portfolio \( h \) and the strategy \( \bar{b}_{T-t} \) replicating \( B_{T-t} \) is a constant weights dynamic strategy given by

\[
\begin{align*}
&\frac{1}{\gamma} \text{ in the numeraire portfolio } H \\
&(1-1/\gamma) \text{ in } B_{T-t} \\
\end{align*}
\tag{15}
\]

- In terms of the four (redundant) assets \( M, S, B_K, B_{T-t} \), the CRRA(\(\gamma,T\)) mutual fund also exhibits constant weights and can be written:

\[
D( (1/\gamma)(1-h_S - h_K), (1/\gamma)h_S, (1/\gamma)h_K, 1-1/\gamma ).
\]
The optimal strategy in terms of the weights in $M$, $S$, and $B_{\kappa}$ is deterministic and given by

$$
\begin{align*}
x_s^* &= \left(1/\gamma\right)h_s \quad \text{in } S \\
x_k^*(t) &= \left(1-1/\gamma\right)\frac{\sigma_{T-t}}{\sigma_k} + \left(1/\gamma\right)h_k \quad \text{in } B_k
\end{align*}
$$

with the complement invested in cash.

The proof of Proposition 1 is given in Appendix A.

As shown in (15), the CRRA($\gamma,T$) mutual fund is a constant weight strategy in two funds, the logarithmic portfolio and the risk-free asset $B_{T-t}$. The weights are the relative risk tolerance $1/\gamma$ on the logarithmic portfolio and the complement $(1-1/\gamma)$ in $B_{T-t}$. Because the logarithmic portfolio has constant weights, when expressed in terms of $B_{K}$ in lieu of $B_{T-t}$, the CRRA($\gamma,T$) mutual fund displays constant weights in the four assets $M$, $S$, $B_{\kappa}$, and $B_{T-t}$. This constant weight strategy is similar to Merton's (1971) result of constant weights in stock and the risk-free asset, but here extended to the case of stochastic interest rates; it involves four assets, depends on the investor horizon $T$ and on the relative risk aversion parameter $\gamma$.

It is useful to compare these results to those obtained in closely connected studies. Sorensen (1999), derived the optimal strategy of a CRRA investor trading three assets ($M$, $S$ and $B_{T-t}$) in a Vasicek bond market, as ours. The expression of the optimum (Sorensen equation (7)), which looks rather different than ours, involves the inverse of the variance-covariance matrix of the returns while our Proposition 1 characterizes the optimal policy as a constant weight strategy. However, the two characterizations would coincide if we expressed the solution in terms of the
sole bond $B_{T,t}$ by synthesizing the bond fund as a dynamic combination of $B_{T,t}$ and cash; such a
synthesis involves a short position on cash (if the horizon $T$ is smaller than the bond fund
duration), a bond weight in the duplicating portfolio tending to infinity as time $t$ gets close to the
horizon $T$, and a cash weight tending to minus infinity. 

This problematic result is explainable though. The optimal portfolio contains the optimal
amounts of stock and interest rate risks. The desired level of stock market risk is obtained by
holding stocks, which also implies some interest rate risk; the desired amount of interest rate risk
is then achieved by an appropriate position in bonds. But the interest rate risk induced by $B_{T,t}$
decreases and tends to zero when $t$ approaches $T$, since the bond volatility (or its interest rate
sensitivity) approaches zero with its time to maturity. In order to obtain the optimal level of
interest rate risk (which is non-zero since the market price of this risk is positive) the investor
must hold amounts of $B_{T,t}$ that increase to infinity as $t$ approaches $T$. In fact, when the sole bond
$B_{T,t}$ is included in the portfolio, it bears two burdens since: (i) it is the asset through which the
portfolio interest rate risk is adjusted to its optimal level at each point of time (between $t$ and
$t+dt$); (ii) it is also the risk free asset over the investment period $(0, T)$. When, in addition to $B_{T,t}$,
the bond fund $B_K$ is also held, it becomes possible to uncouple these two tasks and much
simpler results are obtained. The bond fund picks up the role of adjusting portfolio interest rate
risk leaving the risk-free asset at a constant weight. Indeed, for a CRRA investor the optimal
amount of interest rate risk is constant, provided the market price of risk is constant; moreover
the optimal amount of the risk free asset (over $(0,T)$) is also constant. This is the reason why the
constant weight strategy $D((1/\gamma)(1-h_S-h_K), (1/\gamma)h_S, (1/\gamma)h_K, 1-1/\gamma$) is obtained in our
framework, when the four assets are included in the investment set. All these results can be
viewed as an extension of the standard Merton result: in a framework in which only two assets are traded and where interest rates and risk premiums are constant, it is well known that the optimal weights on stocks and bonds/cash are constant for a CRRA investor.

In fact, in order to compare the theoretical allocations to the popular advice (which involves a bond fund and no $B_{T,t}$), it suffices to formulate the optimal strategies also in terms of the three assets $M$, $S$ and $B_K$ (excluding $B_{T,t}$ from the investment set). Such a formulation (given by equation (16)) can be easily derived by considering the synthesis opposite to the one previously invoked: the duplication of $B_{T,t}$ by a dynamic combination of the bond fund and cash. Since $B_{T,t}$ can be replicated by the deterministic dynamic strategy $b_{T,t}$ involving $M$ and $B_K$ (in proportions $1-\sigma_{T,t}$ and $\sigma_{T,t}$, respectively), it is simple to specify the CRRA($g,T$) mutual fund as a function of the three non-redundant assets $M$, $S$, $B_K$ (equation (16). The weight in stock is constant, as in Merton. The weight on $B_K$ necessary to duplicate $B_{T,t}$ decreases as time passes and the corresponding weight on $M$ increases. Therefore, the overall proportion on $B_K$ decreases (in the reasonable case where $g>1$) and the weight on $M$ increases deterministically as time passes. This rather “mechanical” result does not depend on the stochastic framework assumed for the interest rates. Deelstra, Grasselli and Koehl (2000), obtained an analogous result (proposition 6) in a paper closely related to ours,\textsuperscript{11} where they study the three asset allocation problem in the context of a CIR bond market. Their equation (29), which gives the optimal allocations for a CRRA investor in the CIR case, is comparable to our equation (16), with the differences due to the stochastic environment.

To summarize, in our framework involving stochastic rates, constant risk premiums and three non-redundant assets, the optimal portfolio displays constant weights on four assets ($M$,}
$S, B_K, B_{T_t}$, and in terms of three assets ($M, S, B_K$ or $B_{T_t}$), constant weight in stock and a
deterministic shift of weight from $B_K$ to $M$.

We turn now to the more general case of a HARA utility function $u$ that can be written
as a function of wealth $Y$ as follows:

$$u(Y) = \left( \frac{\gamma}{1 - \gamma} \right) \left( \frac{Y - \hat{Y}}{\gamma} \right)^{1 - \gamma}$$

where $\hat{Y}$ and $\gamma$ are two constants satisfying the following technical condition $(TC)$:

$$(TC) \quad \gamma \text{ and } \hat{Y} \text{ cannot both be } < 0 \quad \text{and} \quad [\hat{Y} B_{T_t}(0) - Y_0] \gamma < 0$$

The domain of definition of the HARA function is $\Delta = \{ Y / (Y - \hat{Y}) \gamma > 0 \}$.

The HARA function displays an absolute risk aversion equal to $\gamma / (Y - \hat{Y})$ which
decreases with wealth if and only if $\gamma > 0$. Two types of HARA functions can then be
distinguished: (1) $\gamma > 0$ which implies decreasing absolute risk aversion, $Y > \hat{Y}$ (from the definition
of $\Delta$), and $\hat{Y} B_{T_t}(0) < Y_0$ (from $(TC)$); (2) $\gamma < 0$ which implies increasing absolute risk aversion,
$Y < \hat{Y}$, and $\hat{Y} B_{T_t}(0) > Y_0$.

For investors with $\gamma > 0$, $\hat{Y}$ can be interpreted as a required lower bound for the terminal
portfolio value ("subsistence level" for which the marginal utility is infinite) and $(TC)$ can be
interpreted as the requirement that the initial investment $Y_0$ should be at least equal to the
discounted value of this lower bound. $\hat{Y} = 0$ is a special case which corresponds to CRRA utility
functions. For investors with $\gamma < 0$, $\hat{Y}$ is an upper bound; as a special case, $\gamma = -1$, $\hat{Y} > 0$
corresponds to a quadratic utility functions with a saturation level $\hat{Y}$. In both cases ($\gamma > 0$ and
$\gamma < 0$) the risk aversion increases with $|\gamma|$. 

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The portfolio optimization of the HARA investor is described in the following proposition:

Proposition 2: Consider the HARA optimization program, assuming (TC)

\[
(PH) \quad \text{Max } E \left[ \left( \frac{\gamma}{1-\gamma} \right) \left( \frac{Y - \hat{Y}}{\gamma} \right)^{1-\gamma} \right] \quad \text{s.t. } E \left[ \frac{Y(T)}{H(T)} \right] = Y_0 \text{ (initial investment)}
\]

(a) The optimal terminal wealth solution of (PH) is

\[
Y^*(T) = k'H(T)^{1/\gamma} + \hat{Y}, \tag{17}
\]

With \(k' = \frac{1}{E(H(T)^{(1/\gamma)-1})} \left[ Y_0 - \hat{Y}B_T(0) \right] \)

\(Y^*(T)\) lies in the domain \(\Delta\) of definition of the HARA utility.

(b) The optimal strategy \((\gamma^*, \hat{Y})\) for the HARA investor (yielding \(Y^*(T)\) solution of (PH)) is a buy and hold combination of two funds:

- the CRRA\(\gamma, T\) fund \(D\left( (1/\gamma)(1-h_K - h_K), (1/\gamma)h_S, (1/\gamma)h_K, 1-1/\gamma) \)
- a zero coupon bond yielding \(\hat{Y}\) at time \(T\).

Proof: (a) The first order conditions (13) are \([ (Y(T) - \hat{Y})/\gamma]^{-\gamma} = \lambda / H(T)\), for some Lagrangian multiplier \(\lambda\), which implies the form (17) of the optimal terminal wealth. The expression for \(k'\) is given by the budget constraint and implies that the sign of \(k'\) is the same than that of \(Y_0 - \hat{Y}B_T(0)\), hence should be positive for investors with \(\gamma>0\) and negative for investors with \(\gamma<0\) which corresponds to the two types of HARA functions described previously.
(b) \(kH(T)^{1/\gamma}\) being the optimal payoff of the CRRA investor with a relative risk aversion \(\gamma\) (see (14)), it follows from (17) that the optimal strategy \((y^*, Y^*)\) for the HARA investor (yielding \(Y'(T)\)) is a buy and hold combination of strategy \((x^*, X^*)\) (the CRRA(\(\gamma, T\) mutual fund) and a zero coupon bond yielding \(\hat{Y}\) at time \(T\).

Propositions 1 and 2 characterize the optimal HARA strategy. This characterization in terms of a buy and hold combination of a zero coupon (maturing at the horizon \(T\)) and a constant weight fund is quite simple. This is due in part to the hypothesis of constant market prices of risk and in part to the artifice of using four assets instead of three for the expression of the strategy.

Proposition 2b implies a static two fund separation \(a la\) Cass and Stiglitz (1970). The two separating funds are the \(T\)-maturity zero-coupon bond and the fund yielding \(H(T)^{1/\gamma}\) at \(T\). Cass and Stiglitz (1970) proved that, in complete markets, a restricted two-fund separation holds for a broad class of utility functions, including the HARA functions.\(^{12}\) Although Cass-Stiglitz two-fund separation is usually applied in a static framework, without consideration of portfolio rebalancing, it is also valid for the analysis of dynamic strategies. With continuous rebalancing, each admissible self-financing strategy can be viewed as a contingent claim; these contingent claims, infinite in number, can be considered as the traded securities of a static framework in which the Cass-Stiglitz separation results apply. This two-fund separation also implies that, contrary to the CRRA investor, the optimal \textit{weights} for the HARA investor are not deterministic, since the ratio of the market values of the two funds follows a stochastic process.
As noted previously, for investors with $\gamma > 0$, $\hat{Y}$ can be interpreted as a lower bound imposed on the terminal portfolio value. The purpose of the zero coupon investment in the optimal combination described in proposition 2(b), is then to ensure that the terminal wealth is higher than this required minimum.

IV. Explaining optimal portfolio behavior

The simple and closed form expressions of the optimal portfolio policies, provided in Propositions 1 and 2, allow a thorough understanding of optimal investment behavior and are helpful for the comparison of this optimal behavior to popular portfolio advice.

The popular advice, as far as strategic asset allocation is concerned, is usually expressed in terms of a dynamic combination of three funds (cash, or money market, stock and bond). Advisors rarely refer to a bond with fixed maturity (and never a bond matching the investors horizon, such as $B_{T-t}$). Consequently, in the following analysis, we include the bond fund $B_K$, but do not explicitly include the zero-coupon bond $B_{T-t}$. In our framework, however, the zero-coupon bond can be synthesized from cash and the bond fund. The following expressions of the optimal asset allocation on $M$, $S$ and $B_K$, in the HARA case derived from proposition 2, are useful:

\[
y^*_s(t) = \left(\frac{w(t)}{g}\right) \frac{hs}{h_K - \frac{y^*_M(t)}{1-y^*_s(t)-y^*_K(t)}}\]

\[
y^*_K(t) = \left(\frac{w(t)}{g}\right) \left(h_K - \frac{y^*_M(t)}{1-y^*_s(t)-y^*_K(t)}\right) + \frac{y^*_M(t)}{1-y^*_s(t)}\]

where $w(t)$ represents the proportion, at time $t$, of the risky fund in the total portfolio value and $1-w(t)=B_{T-t}(t)\hat{Y}/\hat{Y}(t)$, the weight of (the synthetic) $B_{T-t}$. The technical conditions imply that
the signs of ω and γ are the same; The synthetic asset $B_{T-t}$ is a dynamic combination involving $M$ (proportion $1-\sigma_{T-t}/\sigma_K$) and $B_K$ (weight $\sigma_{T-t}/\sigma_K$).

It is now possible to assess and explain the influence on optimal weights of: the investor characteristics, market parameters, stock prices, changes in stock prices and interest rates and the passage of time. Some of the results are known or straightforward; others are far from being obvious when the analysis is not based on a framework yielding closed form solutions and some are subject to recent controversies.

We consider first the effect of the investor’s risk aversion on the bond to stock ratio. Canner, Mankiw and Weil (1997) (CMW in the following) noted that the investment advisors recommend a bond/stock ratio increasing with the investor’s risk aversion. They pointed out an apparent contradiction between this aspect of the popular advice and the standard two fund separation: the latter implies that within the risky fund, hence globally, the ratio bond to stock should be independent with the investor’s risk aversion. By relaxing key assumptions in the simple static theory they were unable to solve the puzzle and suggested that consideration of inter-temporal trading might help resolve the puzzle. Elton and Gruber (2000) have shown that an increase in bond/stock ratio with investor risk aversion can be found in a static mean-variance model for some realistic values of the parameters. They have further established that the popular advice is more easily obtained when short sales are prohibited. Therefore, as this is the most relevant case in a financial planning context, the popular advice as presented in CMW is not necessarily inconsistent with static portfolio theory. Recently, Bajeux-Besnainou, Jordan and Portait (2001), provided an explanation of this puzzle grounded on the continuous time
CRRA model described in the previous section. Their theoretical justification of the popular advice can be extended to any HARA investor for whom the bond to stock ratio, is, from (18):

\[
y^*_K(t)/y^*_S(t) = \left[ h_K - (\sigma_{T,s} / \sigma_K) \right]/h_S + (\sigma_{T,s} / \sigma_K) \omega(t)
\]  

(19)

This ratio increases with risk aversion for all values of the parameters, in complete compliance with this aspect of the popular advice. Brennan and Xia (2001) have also explained the puzzle with a two-factor gaussian model in which both the interest rate and its long-run mean are stochastic.

A second characteristic of the popular investment advice that can be rationalized with this model is that the weight on cash should increase with risk aversion. Indeed equations (18) imply that \( y^*_M \) increases with \( Y \) provided that \( \sigma_K(h_S+h_K) > \sigma_{T,s} \), which is likely to be the case in the situations of the popular advice, since the bond fund is usually taken to be of long duration (such as five years or more) and the growth optimal portfolio is highly aggressive (negative weight in cash, \( h_S+h_K >> 1 \)).

A third important feature of the dynamic strategy that can be analyzed with this model is its so-called convexity. We will call a strategy convex if investors increase (decrease) the weight on stocks as stock prices increase (decrease), along the lines of portfolio insurance. The opposite strategy is called a concave strategy. The following proposition, proved in the appendix B, describes the conditions of convex, versus concave, behavior.

**Proposition 3.** The three following assertions are equivalent:

- The HARA strategy is convex (the weight on the stock fund is increased if its price increases).
• The relative risk aversion decreases with the portfolio value $Y(T)$.

• $\gamma \hat{Y} > 0$.

Proposition 3 establishes that the HARA investor follows a convex strategy if and only if his/her relative risk aversion decreases with wealth. This result can intuitively be understood considering the CRRA case as a benchmark. As the CRRA investor holds fixed proportions of risky stocks it is not surprising that an investor displaying decreasing relative risk aversion increases the proportion invested in risky securities after a price increase that causes an increase in wealth. Proposition 3 states also that the relative risk aversion of a HARA investor decreases with wealth if and only if $\gamma \hat{Y} > 0$. Hence, for investors with $\gamma > 0$, the relative risk aversion decreases (increases) with wealth if and only if $\hat{Y} > 0$. For those with $\gamma < 0$, for whom $\hat{Y} > 0$ (necessary given condition (TC)), the relative risk aversion always increases with wealth.

Consider now an investor with $\gamma > 0$ who requires a minimum terminal portfolio value equal to $\hat{Y}$. If the total wealth of the investor was composed only of the considered portfolio, $\hat{Y}$ would be the “subsistence” level of terminal wealth and should be assumed positive. The investor would thus display decreasing relative aversion and a convex strategy would be optimal. In general, however, the terminal portfolio value $Y(T)$ is only a portion of the total investor’s terminal wealth and therefore $\hat{Y}$ may be negative. In particular, a CRRA investor (considering total wealth) could display a negative value of $\hat{Y}$ if there were other sources of wealth in addition to the considered portfolio. Such a CRRA agent would exhibit constant
relative risk aversion in terms of total wealth and increasing relative risk aversion in terms of the portfolio allocating wealth $Y$.

The previous discussion suggests that an investor with $\gamma > 0$, who cannot afford to loose all of the initial investment in the considered portfolio (therefore with a $\hat{Y} > 0$) displays decreasing relative risk aversion (on the portfolio value) and follows a convex strategy (on the lines of portfolio insurance). On the contrary, when the investor can afford to loose all of the initial investment (the considered portfolio is a small part of the total allocation), the investor will exhibit (apparent) increasing relative risk aversion and will follow a concave strategy.

An investor with $\gamma < 0$ always displays increasing relative risk aversion and always follows a concave strategy. This would be the case of an investor displaying a quadratic utility function (mean-variance preferences, obtained for $\gamma = -1$ and $\hat{Y} > 0$). The case of mean-variance preferences was studied in Bajeux-Besnainou and Portait (1998), Nguyen-Portait (2000) and Bajeux-Besnainou and Jordan (2001).

The following additional comparative statics enhances understanding of other aspects of optimal portfolio behavior.

Consider first the portfolio composition at time 0 and two different values of the interest rate $r(0)$, the other variables and parameters being equal. The weights within the risky fund would be equal in the two cases (the weights of the CRRA strategy depend on risk premia but not on the level of interest rates) while the value of the (synthetic) zero coupon bond yielding a given $\hat{Y}$ at $T$ would be smaller for the higher interest rate (hence the corresponding proportion,
If \( \omega(t) \), would be smaller. Therefore the weights on cash would be smaller and the weight on stocks higher if the required minimum portfolio value \( \hat{Y} \) is positive.

Consider now two different values of the correlation coefficient between the interest rates and the stock returns (two different values of \( \sigma_2 \), the other parameters being identical). Lower correlation might be observed in high-growth, low-inflation economic conditions, when stock values are more dependent on expected future cash flows and less dependent on the (relatively low) riskless interest rate. In lower-growth, higher-inflation regimes, stock values would tend to be more sensitive to changes in interest rates. The growth-optimal portfolio, hence the risky fund, has the same weight in stocks but less weight in bonds and more in cash for the high correlation case (see equations (11) and (16)). The value of the fund yielding \( \hat{Y} \) is identical in the two cases. Overall, the portfolio then contains more cash and a smaller allocation to bonds in the high correlation case. Intuitively, this can be understood as for given market prices of risk \( \lambda_c \) and \( \lambda_r \), the optimal amounts of risks \( z \) and \( z_r \) (i.e. portfolio sensitivities) are given; if for the small \( \sigma_2 \) the weights are at their optimal level, for the high \( \sigma_2 \) the portfolio risk due to \( z_r \) is excessive (while the risk on \( z \) is optimal) and the only way to diminish the \( z_r \) risk without changing the \( z \) risk is to decrease the bond proportion without changing the stock weight. Or, more simply stated, when stocks and bonds are highly correlated, portfolio diversification is increased by holding more cash.

The effect of an unanticipated increase of the interest rates \( (dz_r < 0) \) on the optimal portfolio composition is also interesting. The weights within the risky fund would be unchanged as well as the two components (cash and \( B_k \)) of the synthetic zero-coupon \( B_{T-t} \). However, such
an interest rate increment decreases the value of the two portfolio components, thus triggering a wealth effect which has an impact on the optimal portfolio weights, as stated in proposition 3 (weights in stocks increasing and decreasing in cash, for increasing relative aversion).

V. Conclusion

The model developed in this paper allows dynamic portfolio management to be studied through closed-form solutions in a stochastic interest rate environment for investors displaying HARA utility functions. It is grounded on a continuous-time framework involving three non-redundant assets (stocks, bonds and cash) and a Vasicek-type market to model the stochastic interest rates. We derive first the CRRA solution which is a generalization of the Merton (1971) result of a constant weight strategy on two assets. With stochastic interest rates, the CRRA investor holds the weight in stocks constant while gradually over time re-allocating the remaining wealth from bonds to cash. Then, we derive a simple expression of the optimal strategies for HARA investors; this characterization allows a thorough analysis of the optimal behavior. We prove that the optimal HARA strategies conform to several aspects of the popular advice considered recently by Canner, Mankiw and Weil (1997) and Elton and Gruber (2000). We show that relative risk aversion determines whether HARA investors follow convex or concave strategies. Decreasing relative risk aversion corresponds to convex strategies (similar to portfolio insurance) and increasing relative risk aversion (such as in the mean-variance case) corresponds to concave strategies. We also explain the effects of various differences in market parameters, such as the level of interest rates and the stock-bond return correlation.
Topics for future research include the study of alternative and more general price
dynamics (several state variables), and constraints on investor behavior such as short sale
constraints or minimum terminal wealth when this minimum is not implied by the utility function
but by an institutional constraint.
References


• Markowitz H., 1959, Portfolio selection, New Haven: Yale University Press.


Appendix A

The stochastic structure of the economy and the information flow are represented, as usual, by the probability space \((\Omega, F, P)\) where the filtration \(F\) is generated by the two standard brownian motions \((z, z)\). The measure \(P\) is the historical (or true) probability.

The investment opportunity set is characterized by equations (1) to (5) in the text. A self financing portfolio policy is represented by a bundle such as \((x, X)\), where \(x(t)\) stands for the vector of weights on the risky assets \(S\) and \(B_K(x^* = (x_S, x_B))\) and \(X(t)\) is the corresponding portfolio value at time \(t\); a prime denotes a transpose and an underlined bold letter is a vector. The weight on cash is automatically given by: \(x_C = 1 - x_S - x_B\).

The following proofs, rely on the transformation of the dynamic problem into a static optimization (see Pliska (1986), Cox-Huang (1989) or Karatzas, Lehoczky and Shreve (1987) for the methodology). We also use the fact that the prices of self-financing securities (or portfolios) are martingales under the historical probability \(P\) when the growth-optimal (logarithmic) portfolio \((h, H)\) is used as numeraire. Hence, if \((x, X)\) is self-financing:

\[
\frac{X(0)}{H(0)} = X(0) = E_0 \left\{ \frac{X(T)}{H(T)} \right\}.
\]

Without loss of generality, we set \(H(0) = 1\) and \(E_t\) stands for the conditional expectation.

Finding optimal strategies then implies two steps: first, the determination of the optimal terminal wealth; second, the characterization of the strategies yielding this optimal terminal wealth.

1. **Optimal terminal wealth of the CRRA investor**

The CRRA investor (with a constant relative risk aversion \(\gamma\)) solves:

\[
(P^*) \quad \max_{x(T)} E \left\{ \frac{X(T)^{1-\gamma}}{1-\gamma} \right\} \quad \text{s.t.} \quad E \left\{ \frac{X(T)}{H(T)} \right\} = X_0 = 1 \quad \text{(budget constraint)}
\]

The first order conditions of \((P^*)\) yield the optimal terminal wealth \(X^*(T)\):

\[
(A1) \quad X^*(T) = kH(T)^{1/\gamma}
\]

The parameter \(k\) is determined by the budget constraint

2. **Optimal CRRA strategies**
The optimal strategy yields almost surely (a.s.) the optimal terminal wealth (A1):

Since the market is exactly complete, such a strategy exists and is unique.

The following technical lemma is useful for the derivation of the strategy reaching $H(T)^{1/\gamma}$.

**Lemma 1**

Consider the Ornstein-Uhlenbeck process:

$$dr = a_r (b_r - r) dt - \sigma_r dz_r,$$

$$r(s) = (r_0 - b_r) \exp[-a_r s] + b_r \exp(-a_r s) \int_0^s \exp(a_u) dz_r(u)$$

The following relations hold for $\int_s^T r(s) ds$

(i) $\int_s^T r(s) ds = \alpha(t)(T-t) - \int_s^T \sigma_{T,s}(s) dz_r(s)$, hence:

$$\int_s^T r(s) ds \sim N\left[\alpha(t)(T-t), \eta^2(t)\sigma^2_r(T-t)\right]$$

where:

$$\alpha(t) = b_r + (r(t) - b_r) \frac{\sigma_{T,t}}{\sigma_r(T-t)}$$

$$\eta^2(t) = 1 + 2 \frac{\sigma_{T,t}}{\sigma_T^2} + \frac{\sigma_{2(T-t)}}{2(T-t)\sigma_r}$$

(ii) Also:

$$E\left[(z_r(T) - z_r(t)) \int_s^T r(s) ds\right] = -\delta(t) \sigma_r(T-t)$$

where

$$\delta(t) = \frac{1}{a_r} \left(1 - \frac{\sigma_{T,t}}{\sigma_T(T-t)}\right)$$

**Proof of lemma 1:**

The solution of the Ornstein-Uhlenbeck process is standard and implies that $r(s)$, hence $\int_s^T r(s) ds$, are normally distributed, and

$$r(s) = (r(t) - b_r) \exp[-a_r (s-t)] + b_r \exp(-a_r s) \int_0^s \exp(a_u) dz_r(u)$$

Therefore:
\[
\int_0^T r(s) ds = \left( r(t) - b_t \right) \int_0^T \exp \left[ -a_r (s-t) \right] ds + b_r (T-t) - \sigma_r \int_0^T \exp \left[ -a_r s \right] \exp \left[ a_r u \right] dz_r (u) \\
= \left( r(t) - b_t \right) \frac{1 - \exp \left[ -a_r (T-t) \right]}{a_r} + b_r (T-t) - \sigma_r \int_0^T \exp \left[ a_r (u-s) \right] dz_r (u) ds \\
= \left( r(t) - b_t \right) \frac{1 - \exp \left[ -a_r (T-t) \right]}{a_r} + b_r (T-t) - \sigma_r \int_0^T \frac{1 - \exp \left[ -a_r (T-u) \right]}{a_r} dz_r (u)
\]

which implies (i) since \((1-\exp(-a_r(T-t)))/a_r = \sigma_r/\sigma_r\).

We can also write:

\[
E \left[ \left[ z_r (T) - z_r (t) \right] \int_t^T r(s) ds \right] = -\sigma_r E \left[ \int_t^T dz_r (u) \int_t^T \frac{1 - \exp \left[ -a_r (T-u) \right]}{a_r} dz_r (u) \right] \\
= -\sigma_r \int_t^T \frac{1 - \exp \left[ -a_r (T-u) \right]}{a_r} du \\
= -\sigma_r \left( \frac{1}{a_r} \left[ (T-t) - \frac{1 - \exp \left[ -a_r (T-t) \right]}{a_r} \right] \right) \\
= -\sigma_r \left( T-t \frac{1}{a_r} \left[ 1 - \frac{\sigma_{\tau - t}}{\sigma_r (T-t)} \right] \right)
\]

which is (ii).

It is now possible to describe the strategy yielding \(X^*(T) = kH(T)^{1/\gamma} \) (proposition 1):

**Proof of proposition 1:**

We start from \(h = G^{-1/2} = (SS')^{-1/2} = (S')^{-1/2} \)

where \(S = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ 0 & \sigma_K^2 \end{pmatrix}\) is the diffusion matrix, \(G = SS' = \begin{pmatrix} \sigma_1^2 + \sigma_{12}^2 & \sigma_{12} \sigma_K^2 \\ \sigma_{12} \sigma_K^2 & \sigma_K^4 \end{pmatrix}\) is the variance-covariance matrix of the risky returns \(G^{-1} = \frac{1}{\sigma_1^2 \sigma_K^2} \begin{pmatrix} \sigma_K^2 & -\sigma_{12} \sigma_K^2 \\ -\sigma_{12} \sigma_K^2 & \sigma_1^2 + \sigma_{12}^2 \end{pmatrix}\), \(\theta = (\theta_S, \theta_K)\) are the two risk premia on the risky assets and \(\gamma = S^{-1/2} = (\lambda, \lambda_r)'\) are the market prices of risk.

Then, \(h_S = \frac{\lambda}{\sigma_1}; h_K = \frac{-\lambda \sigma_{12} + \lambda_r \sigma_1}{\sigma \sigma_K}\)

The weights \(h\) are thus constant and the corresponding portfolio value \(H\) obeys:

\[
\frac{dH(t)}{H(t)} = (r(t) + h' \gamma) dt + h' \Sigma dz = (r(t) + \left\| \gamma \right\|^2 dt + \Sigma dz
\]

\(\Sigma = \Sigma = \begin{pmatrix} \sigma_1^2 + \sigma_{12}^2 & \sigma_{12} \sigma_K^2 \\ \sigma_{12} \sigma_K^2 & \sigma_K^4 \end{pmatrix}\)
\[
H(t) = e^{\int_0^t r(u)\,du + \frac{\|z\|^2}{2} (T-t) + \frac{1}{2} \sum_t^T z(t)}
\]

Hence: 
\[
H(T) = H(0)e^{\int_0^T r(u)\,du + \frac{\|z\|^2}{2} (T-t) + \frac{1}{2} \sum_t^T z(t)}
\]

(A3) 
\[
\mathbf{z}' = (z, z_r); \mathbf{z}'' = (\lambda, \lambda_r) \text{ and } \|2\|^2 = (\lambda^2 + \lambda_r^2)
\]

Let \( c = l/\gamma \) and consider the portfolio strategy \((\mathbf{z}, \mathbf{y})\) yielding \( Y(T) \equiv H(T)^c \) a.s.

Since \( \frac{Y(t)}{H(t)} \) is a \( P \)-martingale,

(A4) 
\[
\frac{Y(t)}{H(t)} = E_t\left\{ \frac{Y(T)}{H(T)} \right\} = E_t\left( H(T)^{-c} \right)
\]

which implies that the initial investment required to reach \( H(T)^c \) is \( Y_0 = E_0\left( H(T)^{-c} \right) \).

(A3) and (A4) imply

(A5) 
\[
Y(t) = H(\gamma) E_t\left\{ \exp\left( c - 1 \left[ \int_t^T r(u)\,du + \frac{\|z\|^2}{2} (T-t) + \frac{1}{2} \left( \mathbf{z}(T) - \mathbf{z}(t) \right)^2 \right] \right) \right\} = H(\gamma) E_t\left( e^{N_\gamma} \right)
\]

As \( r \) is gaussian, \( \int r(u)\,du \) is also gaussian and (A3) implies that \( H(T) \) and \( H(T)^{-c} \) are log-normal and that \( N_\gamma \) is normal; then:

(A6) 
\[
Y(t) = H(t)^c e^{E(N_\gamma) + \frac{1}{2} \text{var}(N_\gamma)}
\]

with
\[
E(N_\gamma) = (c - 1) \left[ E_t\left\{ \int_t^T r(u)\,du \right\} + \frac{\|z\|^2}{2} (T-t) \right]
\]
\[
\text{var}(N_\gamma) = (c - 1)^2 \text{var}\left\{ \int_t^T r(u)\,du + \frac{1}{2} \left( z(T) - z(t) \right)^2 \right\}
\]
\[
= (c - 1)^2 \left[ \text{var}\left( \int_t^T r(u)\,du \right) + \frac{\|z\|^2}{2} (T-t) + 2 \text{cov}\left( \int_t^T r(u)\,du, \lambda_r, z_r(T) - z_r(t) \right) \right]
\]

And, using Lemma 1,
Since $\eta(t)$, $\sigma_r$, and $\delta(t)$ are deterministic functions of time, (A6) can be rewritten:

$$Y'(t) = Y(t, H, r) = H'(t) \phi(t) e^{(c-1)\frac{\sigma_r(T-t)}{\sigma_r}}$$

where $\phi(t)$ is a deterministic function of time.

Hence,

$$(A8) \quad \frac{dY(t)}{Y(t)} = c \frac{dH(t)}{H(t)} + (c-1)\frac{\sigma_r(T-t)}{\sigma_r} dt + \left[.\right]dt$$

where $\left[.\right]dt$ stands for a locally deterministic element.

Moreover, since $dr = \left[.\right]dt + \sigma_r dz_r$, (A8) becomes

$$(A9) \quad \frac{dY(t)}{Y(t)} = c \frac{dH(t)}{H(t)} + (c-1)\frac{\sigma_r(T-t)}{\sigma_r} dz_r + \left[.\right]dt$$

Then, since $\frac{dB_K}{B_K} = \left[.\right]dt + \sigma_K dK$, (A9) becomes

$$\frac{dY(t)}{Y(t)} = c \frac{dH(t)}{H(t)} + (c-1)\frac{\sigma_{T-t}}{\sigma_K} \frac{dB_K(t)}{B_K} + \left[.\right]dt$$

which is also the stochastic differential equation governing the self-financing portfolio defined by the weight $c$ on $(h, H)$, and $(c-1)\sigma_r/\sigma_K$ on $B_K$ (the stochastic components of the two self financing strategy returns are identical, hence the drifts must also be equal in absence of arbitrage). Hence, starting from an initial investment $Y_0 = E_0 \left[H(T)^{-1}\right]$, this dynamic combination of $(h, H)$ and $B_K$ reaches $H(T)^c$ a.s. Let $c=1/\gamma$ to obtain equation (15) in proposition 1

The alternative expressions of the optimal CRRA strategy (constant weights on $C$, $S$, $B_K$, $K$, deterministic weights on $S$ and $B_K$ (equation (16)) follow from the previous results v

**Appendix B: Proof of proposition 2- (c)**

Here it is proven that, assuming $(TC)$, the strategy is concave (convex) if and only if $\hat{Y} < 0$ ($\hat{Y} > 0$).
It follows from Proposition 2 (a) and (b), including equation (17), that the optimal portfolio value at any time 
\( t \leq T \) is the sum of its two components and is given by:

\[
Y'(t) = (k'/k)X'(t) + \hat{Y} B_{r,t}(t)
\]

This implies that the optimal stock proportion for the HARA investor is:

\[
y^*_S(t) = \frac{k' X^*_S(t)}{k Y^*_S(t)} y^*_S
\]

which can be written

\[
y^*_S(t) = \frac{x_S^*}{1 + \hat{Y} \frac{k B_{r,t}(t)}{k' X^*_S(t)}}
\]

From Proposition 1, \( k \) is always >0 and from (16), \( x_s^* \) and \( \gamma \) have the same sign.

Assume \( \gamma > 0 \). Recall, from proposition 2, that \( k' = \frac{1}{E[H(T)^{\frac{1}{\gamma-1}}]} \left[ 1 - \hat{Y} B_r(0) \right] \). Condition (TC) implies that 
\( 1 - \hat{Y} B_r(0) > 0 \), hence \( k' > 0 \). From (14), \( X'(T) \) and \( H(T) \) are positively related which implies that when the stock price increases, \( X'(T) \) increases. From (B3) and the signs of all the parameters, \( y^*_S(t) \) increases (decreases) with the stock price when \( \hat{Y} > 0 \) (\( \hat{Y} < 0 \)).

Assume \( \gamma < 0 \) (and \( \hat{Y} > 0 \) from (TC)). Condition (TC) implies \( k' < 0 \). From (16), \( x_s^* \) is negative. From (14), \( X'(T) \) and \( H(T) \) vary in opposite directions. Then (B3) (since (B3) can also be written, using all positive parameters

\[
y^*_S(t) = \frac{-x_S^*}{\hat{Y} k B_{r,t}(t) \left( -k' X^*_S(t) \right)^{-1}}
\]

implies that \( y^*_S(t) \) decreases with the stock price.
The bond and the bond fund are perfectly correlated, thus one is redundant and the model remains in essence a three-assets model.

In theory, at time t the bond fund manager can sell all bonds (with time to maturity K\(\cdot dt\)) and use the proceeds to buy bonds with a time to maturity \(K\cdot dt\). This operation is repeated at \(t+dt\) and so on; then at each point of time this self-financing fund contains only bonds with time to maturity \(K\) and constant volatility \(\sigma_K\). Rutkowski (1999) calls this a rolling horizon bond. In the Vasicek one factor bond market, this \(K\)-maturity bond can also be synthesized by continuously rebalancing a zero-coupon bond of any maturity \(T'\) and the money market instrument. Indeed, a portfolio with a weight \(s_K / s_{T'-t}\) in a zero-coupon with a date of maturity \(T'\) and \(1-s_K / s_{T'-t}\) in cash is a synthetic zero coupon bond with time to maturity \(K\) (the deterministic volatilities \(s_K\) and \(s_{T'-t}\) are defined below in the paper). In practice, the bond fund might include different non-zero-coupon bonds and be managed to maintain approximately a constant duration \(K\). In this case, the price dynamics of the fund would only approximate the constant-maturity zero-coupon bond fund dynamics (equation (4)).

In this case, the Ornstein-Uhlenbeck process is written with a negative sign on the volatility parameter, which is assumed positive. As will become clear later, this assures that all volatilities in the model are positive and induces the negative correlation between interest rates and stock and bond prices which is typically observed.

In the Vasicek model, the risk premium and volatility are constant for a constant-maturity bond fund, but not for a bond maturing at a given date. In the latter case both the risk premium and the volatility decrease as time passes and the bond maturity approaches zero.

\[
\Gamma^{-1} = \frac{1}{\sigma^2} \begin{pmatrix}
\sigma_K^2 & -\sigma_K^2 \\
-\sigma_K^2 & \sigma_1^2 + \sigma_2^2
\end{pmatrix}
\]


The logarithmic portfolio is defined as the portfolio which maximizes the expected logarithmic utility of terminal wealth, \(\max E[\log X(T)] \text{ s.t. } X(0) = 1\).


Since prices change continuously the logarithmic portfolio must be rebalanced continuously to maintain constant weights.

In Sorensen’s equation (7) that gives the optimal weights as a function of the variance-covariance matrix \(\Gamma\), three entries of the latter tend to 0 as \(t\) approaches \(T\) (since \(B(t,T)\), which represents in his notation the volatility of the bond, tends to 0 as \(t\) tends to \(T\)). This implies that the inverse matrix \(\Gamma^{-1}\) (hence the optimal weights) diverges when \(t\) tends to \(T\).

Deelstra, Grasselli and Koehl (2000) extend a previous version of this paper to the case of a CIR framework (for a CRRA investor).

Be aware that in order to use the same two funds for the construction of different optimal portfolios, the parameter \(\gamma\) of the considered investors must be held constant (see, for example Ingersoll p 146 (1987) for this important restriction on the Cass and Stiglitz separation theorems).

The growth optimal portfolio maximizes the expected continuously compounded growth rate of portfolio value subject to no constraint on risk. In the numerical example, we get \(h_S = 139\%\), \(h_K = 72\%\).

The definition of concave strategies is slightly different from the usual “sell after a price increase.” A constant weight strategy implies selling after a price increase purely in order to rebalance the portfolio. We show, however, that the concave strategy derived here is motivated by increasing relative risk aversion. It is also interesting to note that a convex strategy may resemble “momentum” investing and a concave strategy may resemble “contrarian” investing. However, momentum and contrarian investing may be based on beliefs in inefficient markets, which is not considered in our model.

The utility \(u(Y(T))\) induced by this portfolio value must be interpreted, in general, as derived from a more general utility function \(\nu(W(T))\) on the entire wealth \(W(T)\) of the investor. Assume that:
\[
\nu(W(T)) = \left( \frac{\gamma}{1-\gamma} \right) \left( \frac{W(T) - \hat{W}}{\gamma} \right)^{1-\gamma}
\]

where \( \hat{W} \) is the “survival level” of wealth, and that \( W(T) = Y(T) + Z \) where \( Z \) is an exogenous income or wealth, assumed certain for simplicity. Then, the utility derived from the portfolio terminal value should be written:

\[
u(Y) = \left( \frac{\gamma}{1-\gamma} \right) \left( \frac{Y(T) - \hat{Y}}{\gamma} \right)^{1-\gamma}
\]

with \( \hat{Y} = \hat{W} - Z \); \( \hat{Y} \) is the required minimum contribution of the portfolio, which can well be negative.

In the case of a discrete state space \( \Omega \), \((P^*)\) can be written

\[
Max \ L = \sum_{\omega \in \Omega} P(\omega) \left[ \frac{X(\omega)}{1-\gamma} \right]^{1-\gamma} - \kappa \sum_{\omega \in \Omega} \frac{P(\omega) X_T(\omega)}{H_T(\omega)}.
\]

And the first order conditions \( \frac{\partial L}{\partial X_T^{\ast}(\omega)} = 0 \) for all \( \omega \in \Omega \) yield \((B1)\). In the general case \((P^*)\) is a calculus of variations isoparametric problem whose solution relies on the Euler Theorem.