AN ASSET ALLOCATION PUZZLE: COMMENT

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Should the proportions of risky assets in the risky part of an investor's portfolio depend on the investor's risk aversion? According to basic financial theory, in particular the mutual fund separation theorem with a riskless asset, the answer is no. The theorem says that rational investors should divide their assets between a riskless asset and a risky mutual fund, the composition of which is the same for all investors. Risk aversion affects only the allocation between the riskless asset and the fund.

Niko Canner, N. Gregory Mankiw and David N. Weil (CMW) (1997), however, observe that popular investment advice does not conform to this theory. They report the stocks, bonds, and cash allocations recommended by four advisors for conservative, moderate, and aggressive investors. As shown in Table 1, which is reproduced from CMW, the advisors recommend a bond/stock ratio that varies directly with risk aversion.
Table 1. Asset Allocations Recommended by Financial Advisors

<table>
<thead>
<tr>
<th>Advisor and Investor Type</th>
<th>Percent of portfolio</th>
<th>Ratio of Bonds to Stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cash</td>
<td>Bonds</td>
</tr>
<tr>
<td><strong>Fidelity:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conservative</td>
<td>50</td>
<td>30</td>
</tr>
<tr>
<td>Moderate</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Aggressive</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td><strong>Merrill Lynch:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conservative</td>
<td>20</td>
<td>35</td>
</tr>
<tr>
<td>Moderate</td>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>Aggressive</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td><strong>Jane Bryant Quinn:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conservative</td>
<td>50</td>
<td>30</td>
</tr>
<tr>
<td>Moderate</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>Aggressive</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>New York Times:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conservative</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Moderate</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Aggressive</td>
<td>0</td>
<td>20</td>
</tr>
</tbody>
</table>


For example, Fidelity recommends a bond/stock ratio of 1.50 for a "conservative" (more risk averse) investor, a ratio of 1.00 for a "moderate" (less risk averse) investor, and a ratio of 0.46 for an "aggressive" (still less risk averse)
investor. The inconsistency between such advice and the separation theorem is called an asset allocation puzzle by CMW. They attempt to solve the puzzle by relaxing key assumptions in the theory, but finally reach a negative conclusion: "Although we cannot rule out the possibility that popular advice is consistent with some model of rational behavior, we have so far been unable to find such a model." (p.181) However, they suggest that consideration of intertemporal trading might help resolve the puzzle.

In this paper we provide theoretical support for the popular advice. The two key insights are that the investor’s horizon may exceed the maturity of the cash asset and that the investor rebalances the portfolio as time passes. If the investor’s horizon exceeds the maturity of cash, which might be a money market security with maturity of one to six months, then cash is not the riskless asset as is commonly assumed in the basic theory. In a theory allowing portfolio rebalancing, as opposed to a buy-and-hold framework, it is not unreasonable to assume that the investor can synthesize a riskless asset (a zero-coupon bond maturing at the horizon) using a bond fund and cash. Then bonds will be in both the (synthetic) riskless asset and in the risky mutual fund, and we show that in this case the theoretical bond-stock ratio varies directly with risk aversion for any HARA investor.\(^1\) As an example of the type of results that a specific model can produce, we provide a continuous-time model with closed-form solutions which produces theoretical bond-stock ratios similar to the popular advice.

The paper is organized as follows: In the next section, we analyze the advice in terms of the theory of mutual fund separation of David Cass and Joseph E. Stiglitz (1970). We show that this theory is relevant in both static and dynamic frameworks, and use the theory to analyze the popular advice in complete and incomplete markets. In Section II we analyze the advice in the context of Robert C. Merton’s (1971) continuous-time statement of mutual fund separation, and present an illustrative model in which a CRRA investor makes continuous time portfolio decisions under interest rate and stock price uncertainty. In Section III numerical results are compared to the popular advice. Section IV is a conclusion.

I. Cass-Stiglitz Separation, Dynamic Strategies and Investor Horizon

After a discussion of the main features of a framework adequate for the analysis of the puzzle, we provide an explanation of the popular advice grounded on the Cass-Stiglitz (1970) separation theory.
A.\ An Adequate Theoretical Framework for the Analysis of the Popular Advice

Analysis of the popular advice as represented in Table 1 requires careful attention to four aspects of the investment problem: (1) the investor’s horizon, (2) the tradable asset classes included in the popular advice (the “investment basis”), (3) the sources of risk considered and (4) the self-financing strategies (funds) obtainable from the investment basis. First, specifying the investment horizon is necessary for the identification of a riskless asset. Although it is common in buy-and-hold analyses to treat cash as the riskless asset, this implies the investor’s horizon is the same as the maturity of a money market security, perhaps as short as one to six months. Since the popular advice may address investment over several years we make the less restrictive assumption that the investor’s horizon is $T$, which may be equal to or greater than the maturity of cash. Second, since the popular advice involves stocks, bonds and cash, we define the investment basis to consist of a stock fund ($S$), a bond fund ($B_K$) with duration $K$ and money market fund designated as cash ($C$) with maturity $\varepsilon$. Here, a fund’s duration is defined as in John Cox, Jonathan C. Ingersoll and Stephen Ross (1979), namely, that a fund with duration $K$ has the same risk (and return) from interest rate changes as does a zero coupon bond with maturity $K$. Third, a theoretical framework adequate to address the three-asset allocation problem should involve, at least, interest rate risk and stock market risk. Fourth, the popular advisors probably would assume that portfolios will be rebalanced over long horizons, and therefore the analysis should not preclude rebalancing.

Therefore, the most parsimonious framework adapted to an analysis of the popular advice involves three assets, interest rate risk and stock market risk, and allows rebalancing over the investment period. We claim that portfolio theory based on such a framework is compatible with the popular advice.

B.\ A Justification of the Popular Advice Based on Cass-Stiglitz Separation Theory

Cass and Stiglitz (1970) derived the conditions for two-fund separation. In complete markets, two-fund separation holds for a broad class of utility functions, including the HARA functions. In incomplete markets with a riskless asset, two-fund separation holds only for the HARA functions (without a riskless asset, it holds only for quadratic and CRRA preferences). Although Cass-Stiglitz two-fund separation is usually applied in a static framework, without consideration of portfolio rebalancing, it is also valid for the analysis of dynamic strategies.
With continuous rebalancing, each admissible self-financing strategy can be viewed as a contingent claim; these contingent claims, infinite in number, can be considered as the traded securities in a static framework. The Cass-Stiglitz separation results can then be applied in this infinite-claim, static framework.  

When using two fund separation for the analysis of the puzzle, the cases of $T$ equal to $\varepsilon$ and $T$ greater than $\varepsilon$ must be distinguished. It is clear that the puzzle exists when $T$ equals $\varepsilon$. Cash is then the riskless asset (at least when inflation uncertainty is ignored), and two-fund separation holds for the broad class of utility functions in complete markets or for the HARA class in incomplete markets. The two funds are cash and a risky fund consisting of the bond fund and the stock fund. Within the risky fund, the weights are the same regardless of the investor's risk aversion. Theory and the popular advice are indeed inconsistent in this case.

For $T$ greater than $\varepsilon$, we consider first the case of complete markets. The assumption of complete markets would be unrealistic if rebalancing were not allowed. Moreover, in order to have complete markets with an investment basis containing only three assets, no more than two sources of risk, such as interest rate and stock market risk can be considered. If we ignore inflation uncertainty so that interest rate risk means real rate risk, then, for $T > \varepsilon$, the riskless asset is a zero-coupon (nominal) bond ($B_T$) that matures at the investor's horizon. Since this bond does not belong to the investment basis, the investor must synthesize it in a replicating portfolio which is a dynamic combination of $C$ and $B_K$. The risky fund is a dynamic combination of $C$, $B_K$ and $S$. This case is illustrated in Figure 1.

Figure 1 Title: Cass-Stiglitz Two-Fund Separation, Investor Horizon Exceeds Maturity of Cash
Figure 1 Notes:

\[ C = \text{Cash}, \; B_K = \text{Constant K-duration bond fund}, \; B_T = \text{Bond with initial maturity} \]

\((T)\) equal to investor horizon, \(S = \text{Stock fund}\)

<table>
<thead>
<tr>
<th>Synthetic Riskless Fund</th>
<th>Risky Fund</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B_T) replicated with (C, B_K)</td>
<td>(C, B_K, S)</td>
</tr>
</tbody>
</table>

In this situation, the bond/stock ratio varies directly with risk aversion. For example, a less risk averse investor will hold less of the synthetic riskless fund and therefore less of \(B_K\).\(^5\) Since the weights of \(B_k\) and \(S\) within the risky fund do not change, the total bond-stock \((B_k/S)\) ratio will decrease. Therefore, when \(T > \varepsilon\), markets are complete and the investment basis is restricted to consist of the three assets \(C, B_K,\) and \(S\), two-fund separation theory supports the popular advice.\(^5\)

What if markets are incomplete when \(T\) is greater than \(\varepsilon\)? This could be the case if rebalancing were excluded or if additional sources of risk such as inflation uncertainty were included in the framework. In this case the risk free asset cannot be synthesized, and the theory yields ambiguous results. It can be shown numerically, for instance, that the optimal bond-stock ratio for a buy-and-hold investor may vary either directly or inversely with risk aversion depending on the choice of market parameters. Moreover, the results are extremely sensitive to the assumed values of the parameters.\(^7\) Because the theory in the incomplete markets case does not yield a definite prediction about the relationship between risk aversion and the bond-stock ratio, it is not clear that a puzzle exists.\(^8\)
No ambiguity exists, however, if the complete markets analysis is acceptable. The main assumptions that support the complete markets analysis are that rebalancing is allowed, that the bond with a maturity matching the investor’s horizon can be adequately synthesized (with a dynamic combination of cash and the bond fund) and that inflation uncertainty can be ignored. When dynamic strategies are considered, the ability to create contingent claims through rebalancing suggests that the assumption of dynamic completeness may reasonably approximate actual markets (see Darrell Duffie and Chi-fu Huang (1985)). Dynamic strategies are not merely a theoretical construct. They are required in many portfolio applications, such as asset allocation funds, portfolio insurance, and portfolio immunization. The neglect of inflation uncertainty is perhaps more troublesome, especially when considering long horizons, but some empirical evidence (George Pennacchi, 1991) indicates that in the US inflation volatility is smaller than real interest rate volatility. For all these reasons, the complete markets analysis appears reasonable.

II. Continuous Time Analysis

In this section, an additional theoretical justification of the popular advice based on Merton fund separation and an illustrative model are provided.

A. A Justification of the Popular Advice Based on Merton Separation Theory

Within an explicit dynamic framework, the multiple fund separation characterized by Merton (1971) provides an alternative theoretical explanation of the asset allocation puzzle. This approach also allows an analysis of the popular advice in a framework involving intermediate consumption. In a continuous-time economy with \( n \) state variables, all optimal portfolios can be constructed from \( n+2 \) funds. One fund is the instantaneously riskless portfolio. Another fund is the growth-optimal, or logarithmic portfolio, the portfolio which maximizes the expected logarithm of wealth. The \( n \) remaining funds are portfolios with returns perfectly correlated with the state variables. These last \( n \) funds allow hedging against adverse shifts in the investment opportunity set. We consider Merton’s (1973) one-state-variable model \( (n=1) \) which implies complete markets when the investment basis contains three assets. As in Merton, the instantaneous interest rate is taken to be the single state variable driving the investment opportunity set.
In this context the intertemporal consumption-portfolio choice problem of an individual investor may be written

\[
\begin{align*}
\max_{c(t)} & \left[ \int_0^T u(c(t))dt + U(X(T)) \right] \\
\text{subject to} & \quad dX(t) = X(t)(r(t)dt + \mathbf{x}(t)\mathbf{dR}(t) - r(t)\mathbf{dt}) - c(t)\mathbf{dt} \\
X(0) & = X_0
\end{align*}
\]

where \( u \) and \( U \) stand for “well-behaved” utility functions, \( c(t) \) is the consumption stream, \( r(t) \) is the instantaneous risk-free rate, \( \mathbf{x}(t) \) is the vector of weights on risky assets \( S \) and \( B_K \), \( \mathbf{dR}(t) \) is the vector of their instantaneous rates of return, \( \mathbf{1} \) is the unit vector, \( X(t) \) is the wealth process and \( X_0 \) is the initial wealth. Merton (1973) has shown that the solution \( \mathbf{x}^*(t) \) of this consumption-portfolio problem can be written, for any individual, as a dynamic combination of three mutual funds. These three separating mutual funds are the instantaneous risk-free security which we define as \( C \); the growth-optimal portfolio containing \( C, B_K, \) and \( S \); and the hedging portfolio which contains only \( B_K \) since the bond return is perfectly correlated with the state variable\(^{11} \). Figure 2 illustrates this case.

**Figure 2 Title:** Merton Fund Separation

**Figure 2 Notes:**

\[ C = \text{Cash}, \ B_K = \text{Constant K-duration bond fund}, \ B_T = \text{Bond with initial maturity (T) equal to investor horizon}, \ S = \text{Stock fund} \]

**Riskless** | **Growth-Optimal** | **Hedging**
---|---|---
**Fund** | **Fund** | **Fund**
\[ C \] | \[ C \] | \[ B_K \]
\[ B_K \] | \[ B_K \] | \[ S \]
The purpose of the hedging portfolio is to hedge against adverse shifts in the investment opportunity set, which in this model are triggered by decreases in the interest rate (the expected returns of all the securities fall when the interest rate falls thereby worsening investment opportunities). Since a decrease in the interest rate causes an increase in the value of $B_K$, the weight of $B_K$ within the hedging portfolio should be positive. With increasing risk aversion, the weight on the hedging portfolio should increase (increasing $B_K$) and the weight on the growth-optimal portfolio should decrease (decreasing both $B_K$ and $S$). Therefore, the bond-stock ratio will increase. This analysis supports the conclusions derived from Cass-Stiglitz separation.

In addition, it is known that the growth optimal portfolio is highly aggressive and thus levered (negative weight in cash). Increasing risk aversion implies decreasing weight in the growth optimal portfolio and, consequently, increasing weight in cash. This relationship between risk aversion and the cash weight also conforms to the popular advice.

B. An Asset Allocation Model Supporting the Popular Advice

The following one state variable model yields closed form solutions that will be used in Section III to generate numerical results to compare with the popular advice. Markets are assumed arbitrage-free, frictionless, and continuously open between dates 0 and $T$. The investment basis consists of three assets, an instantaneously riskless money market fund with price at $t$ given by $C(t)$; a stock index fund with price $S(t)$; and a bond fund paying $1$ at maturity with price $B_K(t)$ that maintains a constant duration equal to $K$ through continuous rebalancing during the period $0$ to $T$.

The investment opportunity set is a function of one state variable, the instantaneous riskless interest rate $r$ which follows an Ornstein-Uhlenbeck process with constant parameters given by

$$dr(t) = a_r(b_r - r(t))dt - \sigma_r dz_r,$$

where $a_r$, $b_r$ and $\sigma_r$ are positive constants and $dz_r$ is a standard brownian motion. The market price of interest rate risk is assumed constant; thus, this is an Oldrich Vasicek (1977)-type market.

The stochastic processes for the securities are given by
where $\theta_k$ is the risk premium of the bond fund, a constant since the market price of risk is assumed constant and the duration of the fund is maintained as a constant; $\theta_s$ is the risk premium of the stock, also assumed constant; $\sigma_1$, $\sigma_2$ and $\sigma_K$ are positive constants; and $dz$ is a standardized brownian motion orthogonal to $dr$. This model is a simple extension of Black-Scholes and a variant of the Merton one state variable model. It allows for stochastic fluctuations of interest rates which are negatively correlated with bond and stock returns and which induce parallel movements of the expected returns of all the risky assets. The framework implies that the financial markets are dynamically complete.

Portfolio weights are given by $x_i(t), i = C, S, and K$. The corresponding portfolio value is $X(t)$. Our main interest in this paper is the ratio $x_K(0)/x_S(0)$. The choice theoretic problem of an investor with constant relative risk aversion can be stated as the following optimization program:

$$\max_{x(t), i = C, K, S} \mathbb{E} \left[ \frac{X(T)^{1-\gamma}}{1-\gamma} \right]$$

s.t. $X(0) = 1$

$$\frac{dX(t)}{X(t)} = \left( r(t) + x_S(t)\theta_S + x_K(t)\theta_K \right) dt + x_S(t)\sigma_1 dz + (x_S(t)\sigma_2 + x_K(t)\sigma_K) dz_r$$

The solution can be written

$$x_S(t) = \frac{1}{\gamma} h_S$$

(5) $$x_K(t) = \frac{1}{\gamma} h_K + \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_{T-t}}{\sigma_K}$$

and $$x_C(t) = 1 - \frac{1}{\gamma} (h_S + h_K) - \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_{T-t}}{\sigma_K}$$

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The constants $h_S$ and $h_K$ are the weights of the stock index and the bond index, respectively, in the growth-optimal portfolio. The values of the parameters $h_S$, $h_K$, and $\sigma_{T-t}$ can be calculated and are given by

$$h_S = \frac{\lambda}{\sigma_1}; \quad h_K = \frac{-\lambda\sigma_2 + \lambda r\sigma_1}{\sigma_1\sigma_K}; \quad \sigma_{T-t} = \frac{\sigma_f \left( 1 - e^{-\sigma_0(T-t)} \right)}{\sigma_f}$$

with $\lambda$ and $\lambda_r$ implicitly given by

$$\theta_k = \sigma_k \lambda_r \quad \text{and} \quad \theta_S = \sigma_f \lambda + \sigma_2 \lambda_r$$

The optimal weights for the CRRA investor are deterministic functions of time and independent of the level of the interest rate.\(^{17}\)

The optimal bond-stock ratio at any point in time is

$$\frac{x_k(t)}{x_s(t)} = \frac{h_K}{h_S} + (\gamma - 1) \frac{1}{h_S} \frac{\sigma_{T-t}}{\sigma_K}$$

This ratio is increasing in investor risk aversion and thus is consistent with the popular advice. This result holds at any time $t$ and for any value of the parameters. Moreover, the weight in cash conforms to other aspects of the popular advice. Indeed, $x_c(t)$ in equation (5) is decreasing with $\sigma_{T-t}$, hence increasing with $t$, as long as the risk aversion parameter $\gamma$ is greater than 1. Therefore, in conformity with another aspect of the conventional wisdom, the investor will increase the weight in cash as time passes and the horizon gets closer. It can also be shown that the weight in cash increases with risk aversion for all reasonable values of the parameters, in conformity with the popular advice (see section III).

The consistency between this model and the popular advice was predictable since the underlying framework implies that financial markets are complete and the theoretical justification, based on Merton’s separation theory, applies. The analysis has shown that a continuous time portfolio model, grounded on the simplest framework involving stock market risk and interest rate risk, supports theoretically different aspects of the popular advice. It is also useful to see whether the numerical weights produced by this model resemble the weights in the popular advice and to explain any differences that arise.

### III. Comparing Model Weights to the Popular Advice.

Table 2 contains the results for CRRA investors with horizons ranging from three months to ten years.
The bond fund constant duration is $K = 10$ years. The assumed instantaneous interest rate is $r_0 = 4$ percent. The parameters of the Vasicek model, based on K. Chan et al. (1992), are: $a_r = 18$ percent, $b_r = 4$ percent, and $\sigma_r = 2$ percent. The risk premia are assumed to be 6 percent for the stock index and 1.5 percent for a 10-year bond fund; the assumed volatilities are $\sigma_1 = 19$ percent and $\sigma_2 = 6$ percent, hence the stock volatility, $\sigma_s = (\sigma_1^2 + \sigma_2^2)^{0.5}$, is 20 percent and the correlation between stock returns and changes in the short rate, $\sigma_2 / \sigma_s$, is $-0.30$.

Table 2. Asset Allocations for CRRA investors – base case

<table>
<thead>
<tr>
<th>Investor horizon</th>
<th>Investor type and Percent of portfolio</th>
<th>Ratio of Bonds to Stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=3$ months</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>67</td>
<td>15</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>5</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>-14</td>
<td>45</td>
</tr>
<tr>
<td>$T=1$ year</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>55</td>
<td>28</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>5</td>
<td>33</td>
</tr>
<tr>
<td>2</td>
<td>-22</td>
<td>52</td>
</tr>
<tr>
<td>$T=5$ years</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>73</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>5</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-47</td>
<td>78</td>
</tr>
<tr>
<td>$T=10$ years</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-15</td>
<td>98</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>5</td>
<td>-25</td>
</tr>
<tr>
<td>2</td>
<td>-62</td>
<td>92</td>
</tr>
</tbody>
</table>
The bond fund constant duration is $K = 10$ years. The current instantaneous interest rate is $r_0 = 4\%$. The parameters of the interest rate stochastic process (Chan et al. (1992)) are: $a_r = 18\%$, $b_r = 4\%$, and $\sigma_r = 2\%$. The other assumptions are $\theta_K = 1.5\%$, $\theta_S = 6\%$, $\sigma_1 = 19\%$ and $\sigma_2 = 6\%$.

The bond-stock ratios decrease with risk aversion. As expected, only as the horizon decreases toward the maturity of cash, which in this model is infinitesimal, does the ratio approach a constant. For instance, for a one-year horizon, as risk aversion decreases from $\gamma = 8$ to $\gamma = 2$, the ratio decreases from 1.60 to 0.75. In Table 1, the average ratios for the four advisors decrease from 1.20 to 0.25 as risk aversion decreases.

As noted in the theoretical analysis, the model results also correspond to the popular advice in that the weight in cash decreases as risk aversion decreases. For shorter horizons and higher risk aversion, the weight in cash remains positive. For longer horizons and lower risk aversion the investor may lever the portfolio by borrowing cash. This deviation from the popular advice is not surprising for two reasons. First, it is not abnormal that a model which contains no short sale constraints yields levered positions (negative weight in cash) for aggressive investors. Indeed, for long horizons, the T-maturity bond is the riskless asset and cash is a risky asset (with a negative risk premium). For less risk averse investors, the diversification obtained through cash is not sufficient reason to hold such an asset, and a short position in cash is optimal. It is likely that the popular advice, which is intended to apply to all investors regardless of wealth, sophistication and investment experience, does not consider the possibility of levered portfolios. A model with short sale constraints could more easily produce agreement with this aspect of the popular advice. Second, a model that would integrate inflation uncertainty would probably yield higher cash weights, in particular for long horizons. Indeed, rolling over a money market instrument yields a real return that is less sensitive to unexpected changes in inflation than a long-term bond.

Table 2 also shows, for a given investor (value of $\gamma$), how the portfolio weights are modified as time passes. All investors with relative risk aversion exceeding one will increase deterministically the weight in cash as the horizon gets closer. This feature of the optimal portfolio strategy partially conforms to another form of popular advice cited in CMW: As an investor’s horizon gets closer, the investor should switch from risky assets to cash. Because of the CRRA assumption, Table 2 shows a constant weight in stocks at all horizons, holding risk aversion constant. This is a generalization of Merton’s (1973) result of a constant stock weight for CRRA investors when
interest rates are constant and the stock price follows geometric brownian motion. Because of the constant stock weight, in our model the investor reallocates money from bonds to cash as the horizon draws closer.

Table 3 provides some indications of the sensitivity of the results to different parameters for an investor with a five-year horizon.

Table 3. New asset allocations when parameters and assumptions differ from the base case (Table 2) for horizon \( T \) of 5 years.
<table>
<thead>
<tr>
<th>Investor type and percent of portfolio</th>
<th>Ratio of Bonds to Stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor horizon</td>
<td>Cash</td>
</tr>
<tr>
<td>Increase volatility of interest rates ($\sigma$) from 2 percent to 4 percent</td>
<td>8 19 61 19 3.22</td>
</tr>
<tr>
<td></td>
<td>$\gamma$ 5 14 56 31 1.82</td>
</tr>
<tr>
<td></td>
<td>2 -9 33 76 0.43</td>
</tr>
<tr>
<td>Decrease speed of mean reversion ($\alpha_r$) from 18 percent to 10 percent</td>
<td>8 24 58 18 3.14</td>
</tr>
<tr>
<td></td>
<td>$\gamma$ 5 16 55 29 1.87</td>
</tr>
<tr>
<td></td>
<td>2 -17 43 73 0.59</td>
</tr>
<tr>
<td>Change in $b_r$</td>
<td>8 10 73 17 4.18</td>
</tr>
<tr>
<td></td>
<td>$\gamma$ 5 -2 74 28 2.65</td>
</tr>
<tr>
<td></td>
<td>2 -47 78 70 1.12</td>
</tr>
<tr>
<td>Decrease bond fund duration ($K$) from 10 to 5 years</td>
<td>8 -20 102 17 5.87</td>
</tr>
<tr>
<td></td>
<td>$\gamma$ 5 -32 104 28 3.72</td>
</tr>
<tr>
<td></td>
<td>2 -79 109 70 1.57</td>
</tr>
<tr>
<td>Increase bond fund duration ($K$) from 10 to</td>
<td>5 years</td>
</tr>
</tbody>
</table>
Increasing interest rate volatility causes the investor to shift money from the 10-year bond fund into cash and stock, and the bond-stock ratio decreases. This result can be explained by the fact that the instantaneous bond fund return becomes more risky for the same risk premium ($\theta_k=1.5$ percent). At this higher interest rate volatility, the cash weight is positive for all but the most aggressive (here, $\gamma=2$) investor.

Decreasing the speed of mean reversion ($\alpha_r$) is similar to increasing interest rate volatility, because rates are more likely to diverge from the long-run mean. The weights and the bond-stock ratio move in the same direction as in the case of an increase in interest rate volatility. In both cases, leverage is avoided except for the most aggressive investor ($\gamma=2$).

Changing the long-run mean ($b_r$) has no effect. This is linked to the assumption of CRRA utility, which yields portfolio weights that are mainly a function of the risk premia and are independent of the interest rate level.

Since the popular advisors do not specify the duration of the bond fund, the effect of changing the duration of the fund is also worth noting. When the bond fund duration equals the investor horizon, the cash weights are all negative since the investor borrows cash (which is risky) mainly to invest in the now riskless bond fund which yields a higher expected return. In this case, by virtue of the two fund separation principle, the ratio of weights on the two risky assets (cash and stocks) is independent of the investor’s risk aversion (this ratio is approximately equal to $-1.1$ in table 3). When the bond fund duration is extended to 15 years the weights in cash become positive, except for the most aggressive investor.

The previous cases conform to the popular advice except for very aggressive investors with long horizons, for whom cash weights are negative. Even if we presume that the popular advice is not directed towards such investors we might wonder if the model can produce all positive cash weights without requiring unreasonable values of parameters. Table 4 provides an affirmative answer for an investor with a 5-year horizon. All of the cash weights are positive, ranging from 2 percent to 22 percent which is within the range of 0 percent to 50 percent in Table 1.
With few exceptions, the other weights and the bond stock ratios are either within the ranges of Table 1 or not far outside. For example, the “conservative” bond-stock ratio is 1.58 compared to the extreme of 1.50 in Table 1. Only four of the seven parameter values were adjusted to achieve these results: the speed of mean reversion was cut approximately in half, but remains at 10 percent; interest rate volatility was increased by five percentage points; the risk premium on ten-year bonds was reduced from 1.5 percent to 1 percent; and the risk premium on stocks was increased from 6 percent to 8 percent. Changing all seven parameters would likely allow similar results with smaller changes in each parameter, but further analysis along these lines seems superfluous. The sensitivity analysis shows that this model can adequately account for the essential aspects of the popular advice.

IV. Conclusion

Portfolio theory unambiguously supports the popular advice about the allocations to stocks, bonds and cash under two conditions: (1) complete markets and (2) the investor’s horizon exceeds the maturity of cash. Under these two conditions, there is no asset allocation puzzle concerning the changing ratio of the weights of bonds and stocks. Both theory and the popular advice say that the ratio of bonds to stocks should vary directly with investor risk aversion and agree also on other important features of the portfolio strategy. We illustrate the rationality of the popular advice through a continuous time model of stochastic interest rates and stock prices involving one state variable.
References:


HARA means hyperbolic absolute risk aversion. The HARA functions include quadratic utility, which is one way of justifying mean-variance preferences, and constant relative risk aversion (CRRA) utility. Both quadratic and CRRA utility were considered in the CMW analysis.

Two-fund separation was also shown by Fischer Black (1972) for the mean-variance case.

More precisely, consider a standard dynamic continuous-time problem (P) of pure portfolio optimization (without consumption), between 0 and T, the control variables being the portfolio weights \( x \) (the corresponding strategies are constrained to be self-financing):

\[
(P) \quad \max_{x} E[U(X(T))] \quad ; \quad X(0) = X_0
\]

where \( X_0 \) is the initial endowment and \( X(T) \) is terminal wealth. Pliska (1986), Cox and Huang (1989), and Karatzas, Lehoczky and Shreve (1987) have shown that (P) is equivalent to a two step optimization:

**Step 1**: solve \( (P') \)

\[
(P')\quad \max_{X(T)} E(U[X(T)]) \quad \text{s.t.} \quad E^0 \left( \frac{X(T)}{C(T)} \right) = X_0
\]

\( E^0 \) stands for the risk-neutral expectation (taken at time 0) and \( C(T) \) the value at T of $1 invested in cash at time 0.

\( (P') \) yields the optimal terminal wealth \( X^*(T) \), but not the strategy (asset weights) that attains it.

**Step 2**: Find the admissible strategy \( x^*(t) \) that produces the optimal \( X^*(T) \).

From the equivalence between \( (P) \) and \( (P') \) it is clear that the dynamic optimization \( (P) \) (involving continuous rebalancing of a finite number of assets) is equivalent to the static problem \( (P') \) involving an infinite number of contingent claims \( X(T) \) from which the (static) choice must be made.

We assume implicitly that \( B_T \) can be replicated using only \( C \) and \( B_K \). Therefore, not only is the market complete, but the bond market itself can be spanned with only two securities (\( C \) and \( B_K \)). Synthesizing \( B_T \) requires a positive
weight on $B_K$ (positive on $C$ if $K>T$, negative if $K<T$). The replication of a zero-coupon bond maturing at $T$ by a
dynamic combination of fixed-income securities of different durations (in order to obtain at each date $t$ a portfolio
duration of $T-t$) is known among bond portfolio managers as “passive immunization.” See, for example, Gifford
Fong (1990) and Frank J. Fabozzi (1996).

5 It should be noted that there is a restriction on the Cass-Stiglitz two fund separation. When considering different
HARA investors characterized by a marginal utility of wealth, $u'(W) = (A + BW)^c$, where $A$, $B$ and $c$ are utility
parameters and for which risk aversion $= Bc/(A + BW)$, two fund separation holds only if the parameter $c$ is held
constant (see, for example, Jonathan Ingersoll (1987), p. 146).

6 If the riskless bond $B_T$ is included in the investment basis, then the riskless fund consists only of $B_T$ and the risky
fund consists of $C$, $B_K$, and $S$. Within the risky fund, the weights are the same regardless of risk aversion, hence the
ratio $B_K/S$ is the same for all investors. However, the ratio of all bonds ($B_K$ and $B_T$) to stock increases with risk
aversion. Therefore, with this broader definition of the investment basis, it might be claimed that there is no
inconsistency with theory. However, we cannot be sure what the popular advice would be if the investor horizon and
the corresponding zero-coupon bond were part of the advice. We can only say that such advice remains unobserved.

7 See Table A1 in Appendix A on the AER web site.

8 There is a case of incomplete markets in which no puzzle exists. As long as the riskless asset $B_T$ can be synthesized
by dynamic combinations of $C$ and $B_K$, then the analysis provided above in the complete markets case is valid.

There could be any number of sources of risk affecting the stock market which are not hedgeable by the stock fund in
the investment basis. Although the market would then be incomplete, as long as these sources of risk did not prevent
the synthesis of $B_T$ using $C$ and $B_K$, the bond-stock ratio would increase with risk aversion.

9 We thank an anonymous referee for bringing this research to our attention.

10 Merton’s analysis requires no restrictions on the way the risk aversion parameters may vary as in the Cass-Stiglitz
approach (see footnote 5).

11 The prices of all the zero coupon bonds are locally perfectly correlated with the interest rate (the only state
variable).

12 A decrease in the interest rate decreases the expected return on all securities provided that the risk premiums are
constant or, if dependent on the interest rate, that they do not increase enough to offset the effect of the rate decrease.
The growth optimal portfolio maximizes the expected continuously compounded growth rate of portfolio value subject to no constraint on risk.

The model is similar to the one presented in the last section of Isabelle Bajeux-Besnainou and Roland Portait (1998).

An implication of the Vasicek model is that $\sigma_K = \sigma_r \left(1 - e^{-a_r K}\right) / a_r > 0$. The Ornstein-Uhlenbeck process (1) is written with a negative sign on the volatility parameter. Since all volatilities in the model are positive (in particular we assume $\sigma_2$ in (3) is positive, and $\sigma_K$ in (4) is positive according to the previous formula), this negative sign induces the negative correlation between interest rates and stock and bond returns that is typically observed.

See Appendix B available on the AER web site.

These features are specific to CRRA utility functions.

Although there are more recent parameter estimates for the Vasicek model, there does not appear to be a consensus in the literature. For example, Robert R. Bliss and David C. Smith (1998) report a lower rate of mean reversion and lower volatility when the period of the Federal Reserve Board policy change (October 1979-September 1982) is excluded. The parameter values are $a_r = 9.29$ percent, $b_r = 10.65$ percent and $\sigma_r = 1.4$ percent. The model results with these parameters are very similar to the results reported in Table 2.

In Table 4, we show a choice of reasonable assumptions which produces closer agreement with the popular advice, including no leverage.

We thank an anonymous referee for reminding us of the role of cash as an inflation hedge.

Constant stock weights would not be obtained with the more general HARA utility functions. The results are available from the authors.

A lower speed of mean reversion is consistent with the results of Yacine Ait-Sahalia (1996), who finds that interest rates are a random walk within a range of 4 percent to 17 percent. A risk premium for the stock of 8 percent corresponds to the long-term average in the US.