MEAN-VARIANCE ASSET ALLOCATION FOR LONG HORIZONS

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Abstract

We investigate whether mean-variance portfolio theory can produce the conventional wisdom that investors with long horizons should make a large initial allocation to stocks and then decrease the allocation as time passes. For the case of a risk-free asset and a stock index following geometric brownian motion, we derive closed-form solutions for the mean-variance portfolio problem allowing continuous rebalancing based on realized prices and wealth (called a stochastic strategy). This optimal stochastic strategy is in general a conventional wisdom strategy as it involves large initial allocation to stocks which then decreases with time. We relate this strategy to the concave strategies described by Perold and Sharpe and explain the role played by relative risk aversion in this result. We also derive the optimal deterministic strategy (predetermined schedule of weights, independent of new price and wealth realizations) and find it to be a constant-weight strategy.

I. Introduction

The conventional wisdom, as defined in this paper, is that investors with long horizons should allocate most of their liquid wealth to stocks and then decrease the stock allocation as time passes. Not only is such advice very common in the practical investment world (e.g., Del Prete (1997)), but recent survey evidence from the TIAA/CREF pension management organization shows that individual investors behave this way (Bodie and Crane (1997)). As investment in self-directed retirement accounts continues to grow, and as many governments consider social security privatization, it becomes increasingly important to understand the assumptions under which the conventional wisdom is rational.

We ask whether the conventional wisdom is rational in a mean-variance framework. Although the limitations of mean-variance analysis are well established in portfolio theory, its relative simplicity and easy intuition contributes to its continued use among investment professionals, in theoretical and empirical studies and in the classroom. It is of interest to explore the rationality of the conventional wisdom to a mean-variance investor.

Much of the literature on portfolio theory does not support the conventional wisdom. The standard Markowitz problem for two assets (risk-free cash and a stock index following geometric brownian motion) typically dictates a small initial stock allocation for long horizons. Moreover, in this buy-and-hold, or static framework, the stock allocation will typically increase rather than decrease over time due to the positive excess return on stocks. Dynamic portfolio theory provides mixed support for the conventional wisdom. Samuelson (1969) and Merton (1971) show that a constant weight strategy, rather than decreasing stock weights, is optimal if the investor's utility function displays constant relative risk aversion (CRRA) and asset prices follow geometric brownian motion. Samuelson has been able to justify equity weights decreasing over time for CRRA preferences by assuming mean-reverting stock returns (Samuelson (1991)) and by imposing a minimum wealth constraint for geometric brownian motion (Samuelson (1989)), in both cases with a constant interest rate. The literature on turnpike portfolios (e.g., Ross (1974) and more recently Cox and Huang (1992)) shows that for a broad class of utility functions (including the linear risk tolerance, or hyperbolic absolute risk aversion (HARA) functions), the optimal strategy converges to constant weights as the horizon increases.

Bodie, Merton and Samuelson (1992) have certainly found the strongest case for the optimality of the conventional wisdom through the introduction of human capital into the Merton (1971) consumption-investment model. In this model, an initial large equity allocation balances a large, and less risky, initial human capital allocation. Over time, as the human capital allocation declines, so does the equity allocation. A recent additional paper including consumption as well as investment is Campbell and Viceira (1996). The investor is here assumed to be infinitely-lived, and to use an Epstein-Zin-Weil recursive utility function.

We do not consider the consumption-investment problem in this paper, because we wish to focus only on the properties of the mean-variance problem. We consider the stylized problem, similar to the one considered in the turnpike literature, of an investor concerned only about the mean and variance of terminal wealth under self-financing constraints (no intermediate cash flows and consumption).

Two recent papers in a non-mean-variance setting are Brennan (1998) and Barberis (2000). Brennan assumes continuous trading with an uncertain mean return for the risky asset about which the investor learns over time. He finds that allocation to the risky asset depends on the investor's degree of risk aversion. Our paper is similar to the study by Barberis (2000), who investigates three strategies for power utility investors: buy-and-hold, a deterministic strategy with constant weights, and an optimal rebalancing strategyfor a rebalancing interval of one year. Barberis focuses on two issues, parameter uncertainty and predictability in asset returns. We do not deal with these issues because we are focusing in this paper solely on the implications of mean-variance criteria. These other issues could be considered in extensions of this research.

In order to obtain closed-form solutions and useful intuition, we limit the asset allocation problem to two assets, a risk-free asset ("cash") and a stock index. The risk-free rate is constant, and the stock index, which pays no dividends, follows geometric brownian motion. These assumptions are also found in the turnpike literature. In related mean-variance research, Richardson (1989) and Bajeux-Besnainou and Portait (1998) have investigated, in a general setting, optimally rebalanced mean-variance portfolios in continuous time. They have not, however, considered the specific long-horizon issues raised here. Nguyen and Portait (2000) have addressed a very similar mean variance model, but with a solvency constraint, that wealth has to be positive at all times.

In this study, we derive explicitly the optimal stochastic strategy in which portfolio revisions are based on realized prices and wealth. The stochastic strategy shows that the conventional wisdom is, in general, optimal for a mean-variance investor. We note that this stochastic strategy is concave in the stock price (see Perold and Sharpe

(1988)), in that the stock allocation is reduced (increased) after unexpectedly high (low) realized stock prices. Another surprising result is that it produces a wealth distribution which is bounded above and unbounded below, although the probability of negative wealth is almost negligible. We then compare this stochastic optimal strategy with a deterministic strategy defined by an optimal predetermined schedule of weights.

Section II presents the notation and framework and provides the derivation of the optimal stochastic mean-variance strategy. Section III contains numerical examples and interpretations. Section IV is devoted to a comparison with the deterministic optimal strategy. Section V is the conclusion.

II. Optimal Stochastic Portfolio Strategy.

The investment opportunity set consists of two assets at date 0, a risk-free asset (T-bill) with current price B_0 and a risky asset (stock index) with current price S_0. For simplicity, both prices are normalized to one. This allows writing gross returns as B_t and S_t rather than B_t/B_0 and S_t/S_0.

The instantaneous return on the bill is given by

$$\frac{dB_t}{B_t} = rdt \tag{1}$$

where r is the constant instantaneous risk-free rate. For any date $t > 0$, the gross return on the bill is given by e^{rt}. The instantaneous return on the stock index is given by

(2) $$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where μ is the constant instantaneous return, σ is the constant instantaneous standard deviation of return, and W_t is a standard brownian motion.

As in Bajeux-Portait (1998), the mean-variance investor chooses the proportion of initial wealth to allocate to the stock index in order to minimize the variance of terminal ($t=T$) wealth subject to the constraint of desired expected terminal wealth (represented by a desired annualized return α). The stochastic strategy allows the investor to rebalance the portfolio continuously depending on realized prices and wealth. The problem can be stated as optimization program (*P*):

(*P*) $$\begin{aligned} &\min_{x_t} Var(X_T) \\ &\text{s.t. } E(X_T) = e^{\alpha T} \\ &\qquad \frac{\mathrm{dX_t}}{\mathrm{X_t}} = x_t \frac{dS_t}{S_t} + (1 - x_t) \frac{dB_t}{B_t}\ \forall t \end{aligned}$$

where x_t is determined optimally at each date. Note here that, for simplicity, the initial wealth has been normalized to one; thus, X_t is the gross return on wealth at any date t. Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987) have shown that program (*P*) is equivalent in a sense specified below to the following optimization program (*P**) which substitutes a single linear constraint for the self-financing constraints:[1]

[1] This equivalence is obtained under the assumption of complete markets, which is obviously satisfied in our framework containing one brownian motion, two non redundant securities and continuous rebalancing of the portfolio.

$$(P^*) \qquad \begin{aligned} &\min_{X_T} Var(X_T) \\ &\text{s.t. } E(X_T) = e^{\alpha T} \\ &\qquad E^*(X_T) = e^{rT} \end{aligned}$$

where E^* denotes expectation under the risk-neutral probability. The optimization programs are equivalent in the following sense: The solution of program (P) is a portfolio strategy (x_t*); this strategy produces a terminal wealth distribution X_T*; the solution of program (P^*) is the same terminal wealth distribution X_T*. After solving (P^*), the assumption of complete markets is used in deriving the unique portfolio strategy (x_t*) that replicates X_T*.

It is shown in Appendix , using Proposition 1.1 (equation (1.7)) and Lemma 1.2 (equation (1.20)), that the solution to (P*) is given by

$$X_T = \left\{ e^{\alpha T} + k_2 \left(1 - \phi(T)(S_T)^{-\beta} \right) \right\} \tag{3}$$

where

$$k_2 = \frac{e^{\alpha T} - e^{rT}}{e^{\lambda^2 T} - 1} \tag{4}$$

$$\phi(T) = \exp[(\beta\mu - .5\lambda^2 - .5\lambda\sigma)T] \tag{5}$$

and where $\lambda = (\mu\text{-}r)/\sigma$ and $\beta = \lambda/\sigma$.

Wealth at any time can be calculated as the discounted risk-neutral expected value of terminal wealth, that is,

$$X_t = e^{-r(T-t)} E^* \left[X_T \right] \tag{6}$$

Proposition 1.2 with Lemma 1.2 in Appendix 1 leads to

$$X_t = e^{-r(T-t)} \left\{ e^{\alpha T} - k_2 (e^{\lambda^2 (T-t)} - 1) + k_2 e^{\lambda^2 (T-t)} (1 - \phi(t)(S_t)^{-\beta}) \right\} \tag{7}$$

where $\phi(t)$ is given by (5) evaluated at t.

We can clarify (7) by writing it as

(8) $$X_t = E(X_t) + Y_t$$

where the expected value component is given by

(9) $$E[X_t] = e^{-r(T-t)}\left\{e^{\alpha T} - k_2(e^{\lambda^2(T-t)} - 1)\right\}$$

and the stochastic component is given by

(10) $$Y_t = k_2 e^{(\lambda^2 - r)(T-t)}\left\{1 - \phi(t)(S_t)^{-\beta}\right\}$$

The standard deviation of wealth is shown in Proposition 1.2 to be

(11) $$SD\left[X_t\right] = k_2 e^{\left(\lambda^2 - r\right)(T-t)}\sqrt{e^{\lambda^2 t} - 1}$$

Simple calculation (using (4) and (9)) shows that $E[X_t]$ is always positive, and it can be shown that $E[X_t]$ is increasing in t. The stochastic component is an increasing function of the (stochastic) stock price; therefore wealth is an increasing function of the stock price. Since S_t is lognormally distributed (and therefore unbounded above), it can be shown (from (10)) that X_t is bounded above by

(12) $$e^{-r(T-t)}\left\{e^{\alpha T} + k_2\right\}$$

X_t is unbounded below; when S_t goes to zero, X_t goes to negative infinity.

Terminal wealth is a special case of X_t with $t=T$. The expected value of terminal wealth is the objective $e^{\alpha T}$; its upper boundary is $(e^{\alpha T}+k_2)$ and it is unbounded below. It may seem surprising that wealth is possibly infinitely negative. However, low values of wealth occur with very low probabilities. For example, the probability of $X_T < e^{rT}$ is $N[-1.5\lambda\sqrt{T}]$, which is approximately 0.0045 for the parameters assumed in the examplesin Section III below.

Proposition 1.3 in Appendix 1 shows that the optimal allocation to the stock index at any time is given by

(13) $$x_t = \beta\left\{\frac{1}{X_t}e^{-r(T-t)}(k_2 + e^{\alpha T}) - 1\right\}$$

(The proportion x_t is undefined when wealth X_t is zero. Nevertheless, the dollar amounts invested in each security are readily calculated after (13) is multiplied through by X_t.) For the initial allocation, $t=0$, this reduces to

(14) $$x_0 = \beta\left\{e^{-rT}(k_2 + e^{\alpha T}) - 1\right\}$$

The initial allocation in (14) is increasing in the desired expected return α, decreasing in the risk-free rate, and invariant with respect to initial wealth. The subsequent allocation (13) shows that as realized wealth X_t increases (decreases) the portfolio is rebalanced by allocating less (more) to the stock index. This reallocation is necessary to keep the optimal terminal wealth distribution the same after unexpected realizations of the stock price.

III. Comments on the Optimal Stochastic Strategy.

The stock allocation starts initially high and decreases, in general, as time passes. Table 1 provides an example for $T=30$ years, $X_0=\$1$, $r=5\%$, $\mu=12\%$, $\sigma=2\%$ and $\alpha=6.5\%$ (all annual rates), allowing the stock price to evolve along its expected path. The initial allocation is over 85% stock. The allocation to stock decreases (along the average stock price path) to less than 4% at T=30 years. This portfolio strategy has the pattern of the conventional advice.

[Insert Table 1]

The optimality of decreasing the stock allocation on the average stock price path can be explained in terms of investor risk aversion. Problem (P) is isomorphic to the dynamic optimization of the expected utility of terminal wealth for an investor with quadratic utility. The increasing relative risk aversion of the quadratic utility function implies that the increase in investor's wealth, which accompanies the increase in stock prices, causes the investor to allocate a lower proportion of wealth to the risky asset.[2]

We can also study the sensitivity of the initial stock allocation x_0 and the standard deviation of terminal wealth $SD(X_T)$ to changes in the drift and volatility parameters μ and σ. Table 2 summarizes some of these numerical examples.

[Insert Table 2]

Everything being equal, in particular keeping constant the annualized goal return α, an increase (decrease) of the volatility parameter induces a decrease (increase) in the initial stock allocation and an increase (decrease) in the total portfolio risk (standard deviation of final wealth); an increase (decrease) in the drift parameter induces an increase (decrease) in the initial stock allocation and a decrease (increase) in the total portfolio risk. This latter result seems counter-intuitive, as when you increase the drift, you also increase your initial stock allocation. This can be explained by looking, for example, at the total pattern of stock allocation on the average stock path: as the conventional wisdom

[2] Quadratic utility also implies increasing absolute risk aversion and satiation (decreasing utility as wealth increases beyond a certain point). Bajeux-Besnainou, Jordan and Portait (2000) have shown that increasing absolute risk aversion does not generate the decreasing stock allocations, because these levels of wealth are never optimally chosen. These decreasing stock allocations are also found in HARA utility functions, which do not have increasing absolute risk aversion, but which do have increasing relative risk aversion. Satiation is also not a factor in this result, because in problem (P), utility is bounded from above at a level below the satiation level.

is still satisfied (decreasing stock allocation through time), this stock allocation decreases much faster for higher values of the drift (from 118% to 0.3% for a drift of 15% and a volatility of 22%), than for lower values (from 75% to 22% for a drift of 9% and a volatility of 22%).

Perold and Sharpe (1988) have defined concave strategies as strategies for which stock is sold (purchased) as the stock price increases (decreases). The stochastic strategy is a concave strategy requiring rebalancing after unexpected stock price changes. However, along the average stock price path, this strategy rebalances to weights that are decreasing over time. Thus, this strategy will tend to be a more concave strategy than a constant-weight strategy (described in Perold and Sharpe (1988) as a dynamic strategy designed to keep constant weights in all of the different securities). The weight process (x_t) is unbounded above and bounded below by zero. As stock price falls, the investor may find that borrowing at the risk-free rate is necessary to obtain the required allocation to stock. As stock price increases, the allocation will approach zero. As noted above, the resulting wealth process is bounded above and unbounded below.

It is interesting to compare these results in the optimal stochastic case with the results from the Markowitz-type case when no rebalancing is allowed between over the investor's horizon. Any reasonable values of the parameters of the T-bill and stock price processes result in an optimal solution of small initial stock allocations which increase over time as the stock price increases faster than the T-bill price over most stock paths. If we rerun the numerical example with the previous set of parameter values as in Table 1, by $t = 30$, the portfolio standard deviation is 525% compared to 57% with the stochastic rebalancing schedule. In the buy and hold strategy, only by choosing a relatively small initial allocation to the stock index (the example would provide a 7.83% initial stock allocation) can the investor minimize the variance of terminal wealth. When stochastic rebalancing as described in section II is allowed, such conservatism is unnecessary because the stock allocation can be reduced (on average) over time to keep the ending variance within desired bounds.

Our results imply that for mean-variance investors, large gains in risk control are obtainable through continuous rebalancing. Of course, the application of these results would be limited by transaction costs. An area for future research is to investigate the practicality of a rebalancing approach given realistic transaction costs. Similar exercises in the literature include studies of the continuous rebalancing assumption in option pricing and dynamic hedging, such as Mercurio and Vorst (1996) and Loewenstein (2000). The general results of such studies is that continuous time models are approximations of varying degrees of accuracy depending on how the much the frequency of trading must be reduced in order to optimally trade off transaction costs and tracking error. However, as a practical matter, option pricing and dynamic hedging strategies (such as portfolio insurance) continue to be widely applied.

IV. Comparison with a deterministic optimal schedule.

The conventional advice might also be interpreted as recommending a pre-determined rebalancing schedule. For example, the Fidelity Investments organization (Del Prete (1997)) has recommended a schedule of allocations for investors with horizons of more than ten years (100% stocks), seven to ten years (70% stocks), four to seven years (50% stocks), two to four years (20% stocks), and less than two years (no stock). Similar "rules of thumb" are common. Barberis (2000) explores a similar idea; he defines and investigates a strategy of “myopic rebalancing,” in which the same predetermined allocation is reset at the start of every year. We pose a more general question: What is the optimal predetermined schedule of allocations? Note that in our continuous-time setting, such strategies imply continuous rebalancing: since x_t is constrained to have the same value whatever the value of S_t, the investor almost surely will have to rebalance the portfolio.

The problem can be stated as problem (P_D):

$$(P_D) \qquad \begin{aligned} &\underset{x_t}{Min} \quad Var(X_T) \\ &s.t. \quad E(X_T) = e^{\alpha T} \\ &\qquad \frac{\mathrm{dX_t}}{\mathrm{X_t}} = x_t \frac{dS_t}{S_t} + (1 - x_t) \frac{dB_t}{B_t} \quad \forall t \end{aligned}$$

where x_t is a deterministic function of time. The additional constraints are the self-financing constraints required in a model with rebalancing. The derivation of the optimal deterministic weights is provided in Appendix 2; in particular, Proposition 2.4 shows that the solution is a constant proportion strategy, defined by a constant weight x in the stock index:

$$x = \frac{\alpha - r}{\mu - r} \tag{15}$$

From this result, we can conclude that an investment plan recommending a predetermined strategy of decreasing weights is a suboptimal mean-variance strategy. In addition, this result provides a new case of the optimality of constant weight strategies that does not depend on CRRA utility with a finite horizon (as in Merton (1971)) nor on more general utility functions with an infinite horizon as in the turnpike literature.

The results of the deterministic portfolio strategy are illustrated in Table 3 for the same parameters as in the optimal stochastic case.

[Insert Table 3]

The expected terminal wealth is the same for both strategies because it is imposed by constraint. The standard deviation of terminal wealth of 185% is more than three times the standard deviation in the optimal strategy of 57%. However, it is about 1/3 the buy-and-hold standard deviation of 525%. With this restrictive method of predetermined rebalancing, the investor has much less control of portfolio risk than an investor following the optimal stochastic strategy.

V. Conclusion

The conventional wisdom about long-term investing is to start with large stock weights which then decrease over time. In this paper, we ask whether such a strategy is optimal for a mean-variance investor with a long horizon. We derive closed-form solutions for two dynamic mean-variance strategies in a continuous-time, complete markets framework.

In the optimal strategy, the allocations each period depend on realized prices and wealth. Only this strategy entails large initial weights that decrease (on the average stock price path) over time, an allocation resembling the usual advice about investing for the long term. The ability to optimally rebalance the portfolio as time passes and wealth is determined allows the investor to take large initial positions without getting locked into large end-of-horizon portfolio risk. The optimality of decreasing stock allocations over time on the average stock price path is due to the close relationship between the mean-variance minimization problem and the problem of maximizing quadratic utility. The increasing relative risk aversion of the quadratic utility function implies that the investor reduces the allocation to the risky asset as wealth increases[3].

This research reveals some general characteristics of an optimal stochastic strategy. It is a concave strategy in that the investor's stock allocation is concave in the stock price. Wealth is bounded above and unbounded below. This seems counterintuitive given that the lognormal stock price distribution is bounded below and unbounded above. The counterintuitive result is a consequence of continually reducing the stock allocation when wealth gets too high and increasing the stock allocation when wealth gets too low in order to meet the expected terminal wealth constraint.

The second strategy investigated is a deterministic strategy in which a predetermined schedule of portfolio weights is defined and not adjusted over time for realized prices and wealth. Given that an investor would consider such as strategy (as considered, for example, in Barberis (2000)), we show that the optimal continuously-rebalanced deterministic strategy is a constant-weight strategy. Any investment plan recommending a predetermined strategy of decreasing weights is a suboptimal mean-variance strategy, at least in the case of geometric brownian motion with constant parameters.

Possible extensions of this research include parameter uncertainty, predictability in asset returns and the effects of transaction costs.

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[3] At the suggestion of an anonymous referee, we point out that our theoretical support for the conventional wisdom does not preclude other explanations, including frequent, but sub-optimal rebalancing due to adverse short horizon portfolio managers' incentives.

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APPENDIX 1

Lemma 1.1: The Radon-Nikodym derivative at date 0, Z_T, defining the change of probability from the original to the risk-neutral probability, satisfies the following dynamics: $dZ_t/Z_t = -\lambda dW_t$, where $\lambda = (\mu-r)/\sigma$ is the instantaneous price of risk. Then,

$$Z_T = e^{-\frac{1}{2}\lambda^2 T} e^{-\lambda W_T} \tag{1.1}$$

and

$$Z_t = e^{-\frac{1}{2}\lambda^2 t} e^{-\lambda W_t} \tag{1.2}$$

Defining now Z_{T-t} as the Radon-Nikodym derivative at date t of the change in probability we get

$$Z_{T-t} = e^{-\frac{1}{2}\lambda^2 (T-t)} e^{-\lambda(W_T - W_t)} \tag{1.3}$$

Then,

$$Z_T = Z_t Z_{T-t} \tag{1.4}$$

$$E_t\left(Z_{T-t}\right) = 1 \tag{1.5}$$

and

$$E_t(Z_{T-t}{}^2) = e^{\lambda^2 (T-t)} \tag{1.6}$$

Proposition 1.1: The risk-neutral terminal wealth distribution that solves Problem (P^*) is

$$X_T = [e^{\alpha T} + k_2(1 - Z_T)] \tag{1.7}$$

where

$$k_2 = \frac{e^{\alpha T} - e^{rT}}{e^{\lambda^2 T} - 1} \tag{1.8}$$

Proof of Proposition 1.1: Using the Radon-Nikodym derivative, the second constraint in Problem ($P*$) can be written in terms of the actual probability,

$$E(X_T Z_T) = e^{rT}$$

The Lagrangian for Problem ($P*$) is then[4]

$$L = E[X_T(\omega)^2] - 2k_1\{E[X_T(\omega)] - e^{\alpha T}\} + 2k_2\{E[X_T(\omega)Z_T(\omega)] - e^{rT}\} \qquad (1.9)$$

where for mathematical convenience the first Lagrangian multiplier is defined to be ($-2k_1$), and the second Lagrangian multiplier is defined to be ($2k_2$). In (1.9) the dependence of X_T on the state of nature ω is made explicit by writing $X_T(\omega)$. The first order conditions can be written

$$X_T(\omega) = k_1 - k_2 Z_T(\omega) = 0 \qquad (1.10)$$

$$E[X_T(\omega)] = e^{\alpha T} \qquad (1.11)$$

$$E[X_T(\omega)Z_T(\omega)] = e^{rT} \qquad (1.12)$$

Evaluating the expectation $E(X_T)$ from (1.10) and substituting into (1.11) produces

$$k_1 - k_2 = e^{\alpha T} \qquad (1.13)$$

Substituting (1.10) into (1.12) and simplifying produces

$$k_1 - k_2 e^{\lambda^2 T} = e^{rT} \qquad (1.14)$$

where in the simplification we use (2.6) with $t=0$. From (1.13) and (1.14) we have (1.8). From (1.8), (1.10) and (1.13) we have (1.7).

Proposition 1.2: The wealth at time t is given by

$$X_t = e^{-r(T-t)}\left\{e^{\alpha T} - k_2(e^{\lambda^2(T-t)} - 1) + k_2 e^{\lambda^2(T-t)}(1 - Z_t)\right\} \qquad (1.15)$$

Its expected value and standard deviation are given by

$$E[X_t] = e^{-r(T-t)}\left\{e^{\alpha T} - k_2(e^{\lambda^2(T-t)} - 1)\right\}$$

$$SD[X_t] = k_2 e^{(\lambda^2 - r)(T-t)}\sqrt{e^{\lambda^2 t} - 1}$$

<u>Proof of Proposition 1.2</u>:

Using the risk-neutral probabilities, wealth at time t is the discounted risk-neutral expected value of terminal wealth, conditional on information at time *t*,

$$X_t = e^{-r(T-t)}E_t^*(X_T) = e^{-r(T-t)}E_t(X_T Z_{T-t}) \tag{1.16}$$

where E_t is the conditional expectations operator. Substituting from (1.7) for X_T produces

$$X_t = e^{-r(T-t)}E\left[(e^{\alpha T} + k_2(1 - Z_T))Z_{T-t}\right] \tag{1.17}$$

Using (1.4) and (1.5) we have

$$X_t = e^{-r(T-t)}\left\{e^{\alpha T} + k_2 - k_2 Z_t E(Z_{T-t}{}^2)\right\} \tag{1.18}$$

Using (1.6) this can be written

$$X_t = e^{-r(T-t)}\left\{e^{\alpha T} + k_2(1 - e^{\lambda^2(T-t)}) + k_2 e^{\lambda^2(T-t)}(1 - Z_t)\right\} \tag{1.19}$$

The computation of expected values and standard deviation are derived from the previous expression and (1.5) and (1.6) from lemma 1.1.

<u>Lemma 1.2</u>: The Radon-Nikodym derivative can be written in terms of the stock price process as

$$Z_t = \phi(t)(S_t)^{-\beta} \tag{1.20}$$

where

$$\phi(t) = \exp[(\beta\mu - .5\lambda^2 - .5\lambda\sigma)t] \tag{1.21}$$

[4] Minimizing $E[X_T^2]$ is equivalent to minimizing $Var[X_T]$ because $E[X_T]$ is a constant, $X_0 e^{\alpha T}$.

and $\beta = \lambda/\sigma$.

Proof of Lemma 1.2: From the assumed stochastic process for the stock index, we have

$$S_t = e^{\mu t - \frac{1}{2}\sigma^2 t} e^{\sigma W_t} \tag{1.22}$$

or

$$e^{\sigma W_t} = S_t e^{(\frac{1}{2}\sigma^2 - \mu)t} \tag{1.23}$$

From (1.2),

$$Z_t = e^{-\frac{1}{2}\lambda^2 t} e^{-\lambda W_t} = e^{-\frac{1}{2}\lambda^2 t} (e^{\sigma W_t})^{-\beta} \tag{1.24}$$

Substituting (1.23) into (1.24) proves Lemma 2.

Proposition 1.3: The allocation to the stock index as a proportion of initial wealth is given by:

$$x_t = \beta \left\{ \frac{1}{X_t} e^{-r(T-t)} (k_2 + e^{\alpha T}) - 1 \right\} \tag{1.25}$$

Proof of Proposition 1.3: First, from (1.19), the dynamics of X_t are given by

$$dX_t = [\cdot]dt - k_2 e^{(\lambda^2 - r)(T-t)} dZ_t \tag{1.26}$$

(The drift term is not needed explicitly for the proof of this Proposition.) Substituting $dZ_t/Z_t = -\lambda dW_t$ into (1.21) produces

$$dX_t = [\cdot]dt + \lambda k_2 Z_t e^{(\lambda^2 - r)(T-t)} dW_t \tag{1.27}$$

(1.19) can also be written

$$k_2 e^{(\lambda^2 - r)(T-t)} Z_t = e^{-r(T-t)} (e^{\alpha T} + k_2) - X_t \tag{1.28}$$

Substituting (1.28) into (1.27) yields

$$dX_t = [\cdot\cdot]dt + \lambda\left\{e^{-r(T-t)}(e^{\alpha T} + k_2) - X_t\right\}dW_t \tag{1.29}$$

Second, it is also true that the process for X_t can be written

$$\frac{dX_t}{X_t} = x_t \frac{dS_t}{S_t} + (1 - x_t)rdt \tag{1.30}$$

where x_t is the proportion of market value of the portfolio allocated to the stock index.

(1.30) can also be written

$$dX_t = X_t(r + x_t(\mu - r))dt + \sigma x_t X_t dW_t \tag{1.31}$$

Equating the volatility terms in (1.29) and (1.31) implies

$$\sigma x_t X_t = \lambda\left\{e^{-r(T-t)}(e^{\alpha T} + k_2) - X_t\right\} \tag{1.32}$$

This can be written

$$x_t = \beta\left(\frac{1}{X_t} e^{-r(T-t)}(e^{\alpha T} + k_2) - 1\right) \tag{1.33}$$

APPENDIX 2

<u>Proposition 2.1</u>: The wealth process follows

$$\frac{dX_t}{X_t} = \alpha_t dt + \sigma_t dW_t \tag{2.1}$$

where

$$\alpha_t = r + x_t(\mu - r) \text{ and } \sigma_t = \sigma x_t \tag{2.2}$$

Terminal wealth is given by

$$X_T = \exp\left[\int_0^T \alpha_t dt - (1/2)\int_0^T \sigma_t^2 dt\right] e^{\int_0^T \sigma_t^2 dW_t} \tag{2.3}$$

<u>Proof of Proposition 2.1</u>: The stochastic process for X_t is given by

$$\frac{dX_t}{X_t} = x_t \frac{dS_t}{S_t} + (1 - x_t)\frac{dB_t}{B_t} \tag{2.4}$$

Substituting (1) and (2) yields

$$\frac{dX_t}{X_t} = \left[r + x_t(\mu - r)\right]dt + \sigma x_t dW_t \tag{2.5}$$

Substitution of (2.2) into (2.5) and application of Ito's Lemma yields the proposition. □

<u>Proposition 2.2</u>: $E\left[X_T\right] = e^{\int_0^T \alpha_t dt}$ (2.6)

<u>Proof of Proposition 2.2</u>: Taking the expectation of (2.3), and requiring that x_t, and thus α_t and σ_t, be deterministic, yields

$$E\left[X_T\right] = \exp\left[\int_0^T \alpha_t dt - (1/2)\int_0^T \sigma_t^2 dt\right] E\left[e^{\int_0^T \sigma_t dW_t}\right] \tag{2.7}$$

The distribution of $\int_0^T \sigma_t dW_t$ is normal with zero mean and variance $\int_0^T \sigma_t^2 dt$. Therefore it can be written as a linear function of the standardized normal variable,

$$\int_0^T \sigma_t dW_t = \left(\sqrt{\int_0^T \sigma_t^{\,2} dt}\right) n \qquad (2.8)$$

where $n \sim N(0,1)$. Therefore,

$$E\left[e^{\int_0^T \sigma_t dW_t}\right] = E\left[e^{\left(\sqrt{\int_0^T \sigma_t^{\,2} dt}\right) n}\right] \qquad (2.9)$$

It is well-known that for a constant *s*, $E\left[e^{sn}\right] = E\left[e^{(1/2)s^2}\right]$. Therefore,

$$E\left[e^{\int_0^T \sigma_t dW_t}\right] = E\left[e^{(1/2)\int_0^T \sigma_t^{\,2} dt}\right] \qquad (2.10)$$

Substitution into (2.7) yields the proposition. □

Proposition 2.3: $E\left[X_T^{\,2}\right] = \exp\left[2\int_0^T \alpha_t dt + \int_0^T \sigma_t^{\,2} dt\right]$ (2.11)

Proof of Proposition 2.3: The expectation of X_T^2 is

$$E\left[X_T^{\,2}\right] = \exp\left[\int_0^T \alpha_t dt - \int_0^T \sigma_t^{\,2} dt\right] E\left[e^{2\int_0^T \sigma_t dW_t}\right] \qquad (2.12)$$

From here the proof proceeds as in Proposition 2.2. □

Proposition 2.4: The solution of Problem (P_U) is

$$\alpha_t = \alpha \qquad (2.13)$$

Thus,

$$x_t = \frac{\alpha - r}{\mu - r} = x \qquad (2.14)$$

Proof of Proposition 2.4: Using Proposition (2.2), Problem (*P*) can be written

$$\underset{x_t}{Min}\ E(X_T^{\,2})$$
$$s.t.\ \mathrm{e}^{\int_0^{\mathrm{T}} \alpha_t dt} = e^{\alpha T}$$

Minimizing this objective function is equivalent to minimizing the variance of X_T because $E(X_T)$ is constrained to be a constant in Problem (*P*). The new constraint is derived from the previous one by substituting for $E(X_T)$ from Proposition (2.2). Substituting (2.11) for the objective function and using the constraint results in

$$\underset{x_t}{Min}\ e^{\int_0^T \sigma_t^2 dt}$$
$$s.t.\ \int_0^T \alpha_t dt = \alpha T$$

Because r, μ, and σ are constants, from (2.2) this optimization program has the same solution as the following problem (P'):

$$\underset{x_t}{Min}\ \int_0^T \alpha_t^{\,2} dt \qquad \text{(P')}$$
$$s.t.\ \int_0^T \alpha_t dt = \alpha T$$

Define $y_t = \alpha_t - \alpha$. Then from the constraint in (P'),

$$\int_0^T y_t dt = \int_0^T \alpha_t dt - \alpha T = 0 \qquad (2.15)$$

The objective function in (P') can be written

$$\int_0^T \alpha_t^{\,2} dt = \int_0^T y_t^{\,2} dt + 2\alpha \int_0^T y_t dt + \alpha^2 \qquad (2.14)$$

of which the last two terms on the right side are constants. Therefore, problem (P') can be written as problem (P''),

$$\underset{x_t}{Min}\ \int_0^T y_t^{\,2} dt \qquad \text{(P'')}$$
$$s.t.\ \int_0^T y_t dt = 0$$

The solution can be written by inspection. The objective function is a sum of quadratic terms. The minimum value occurs at $y_t = 0$. From the definition of y_t and equation (2.2) we have the proof of the proposition.□

Table 1

Mean-variance model and optimal portfolio selection with continuous stochastic rebalancing on the average price path.

Assumed parameters:

r= 5%; μ=12%; σ = 22%; α = 6.5%; T = 30

Calculated parameters:

λ = 31.82%; β = 1.45; k_2 = 0.13

Year t	B_t	$E(S_t)$	$E(X_t)$	$SD(X_t)$	$x(t)$ **on** $E(S_t)$	X_t **on** $E(S_t)$
0	\$1.00	\$1.00	\$1.00	0%	86.34%	\$1.00
1	\$1.05	\$1.13	\$1.11	19%	75.20%	\$1.10
5	\$1.28	\$1.82	\$1.59	38%	45.65%	\$1.56
10	\$1.65	\$3.32	\$2.28	47%	26.33%	\$2.23
15	\$2.12	\$6.05	\$3.10	52%	15.86%	\$3.05
20	\$2.72	\$11.02	\$4.13	55%	9.80%	\$4.07
25	\$3.49	\$20.09	\$5.41	56%	6.14%	\$5.35
30	\$4.48	\$36.60	\$6.67	57%	3.88%	\$6.97

Table 2

Initial stock allocation and standard deviation of final wealth for different drift and volatility values.

<u>Assumed parameters:</u>

r= 5%; α = 6.5%; T = 30

σ / μ	9%		12%		15%	
	x_0	$SD(X_T)$	x_0	$SD(X_T)$	x_0	$SD(X_T)$
18%	91%	138%	124%	26%	175%	2%
22%	75%	196%	86%	57%	118%	11%
26%	66%	250%	66%	91%	85%	28%

Table 3

Mean-variance model and optimal portfolio selection in the deterministic case.

Assumed parameters:

r = 5%; μ=12%; σ = 22%; α = 6.5%; T = 30

Year t	B_t	$E(S_t)$	$E(X_t)$	$SD(X_t)$	x(t) on $E(S_t)$	X_t on $E(S_t)$
0	$1.00	$1.00	$1.00	0%	21.43%	$1.00
1	$1.05	$1.13	$1.07	5%	21.43%	$1.07
5	$1.28	$1.82	$1.38	15%	21.43%	$1.41
10	$1.65	$3.32	$1.92	29%	21.43%	$2.00
15	$2.12	$6.05	$2.65	49%	21.43%	$2.82
20	$2.72	$11.02	$3.67	78%	21.43%	$3.98
25	$3.49	$20.09	$5.08	121%	21.43%	$5.62
30	$4.48	$36.60	$7.03	185%	21.43%	$7.94