Optimal Sequential Auctions for Complements

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Abstract

A sequence of optimal auction mechanisms is derived in a framework where two complementary objects become available over time. With independent revenue-maximizing sellers, the sequence of optimal auctions may allocate the two objects as a bundle either more or less often than is socially efficient. Moreover, each seller imposes a negative externality on the other seller by maximizing own revenue. Consequently, a sequence of coordinated auction mechanisms can increase joint revenue. Under the joint optimal auction mechanism objects are allocated as a bundle more often than under independently designed auction mechanisms.

Keywords: Optimal auctions, sequential auctions, complementarities

1 Introduction

Many auctions are recurring events where bidders may reasonably expect to meet current opponents again in a future auction under terms that are affected by the outcome of previous auctions. In many such instances bidders who have won a previous auction experience an 'increase' of their random valuation in subsequent auctions, due to complementarities, scale economies, learning effects etc. Many examples can be given along these lines:

- A contract to develop and supply a new generation fighter jet is auctioned recurrently. Experience and investment in production technology gives a competitive advantage relative to entrants. Consequently, winning several auctions over time is more valuable than the sum of winning each auction alone.

- Monopoly profit is generally larger than aggregate oligopoly profits in a market. Sequential auctions of licenses to operate in a market provides an example where the complementarity is due to an increase in market power from owning more than one object.

- UMTS spectrum licences were auctioned in sequence across many European countries. Winning a licence in more than one country implies that not all investments in developing networks etc. has to be duplicated. Also, scale economies in R&D for developing services are likely to be present, hence a bundle of licences is more valuable than the sum values of each licence alone.

The essential characteristics of these examples are that several units are auctioned recurrently, after some laps of time, and that bidders' valuations are stochastically dependent across auctions, due to learning, scale economies, market power etc. The aim of this paper is to gain insight into how complementarities may affect the problem of optimal auction design when objects become available over time. A simultaneous auction mechanism as studied by J. Levin (1997) is thus ruled out a priori. J. Levin considers the optimal selling mechanism for complementary objects when bidders know the (identical) value of both objects at the outset and simultaneous sales are assumed. The present paper deviates from that paper by enforcing a sequential sales mechanism. Moreover, the value of each object is only known prior to each auction.

While the presence of complementarities in itself may lead to inefficiencies, the sequential nature of the problem does not make things better. Krishna (1993) gives an example where additional scarce resources become available over time in a duopoly, and argues, that sequential auctions lead to the benefits of aggregation not being realized.

Other papers analyze the consequences of auctioning complementary objects in a sequence, however, not by means of optimal auction mechanisms. F. Branco (1997), F. Menezes and P. Monteiro (1999) and Chapter 1 of this thesis show that sequential second-price auctions of complementary objects offers an explanation on the declining price anomaly first identified by O. Ashenfelter (1989). An important assumption in those papers is that all bidders are treated symmetrically, i.e. auction designs that discriminates among winners of earlier auctions, and bidders that did not acquire an object, are not considered. The intuition behind the result of declining expected prices is that the winner of the first auction is allowed to enjoy the benefit from being a 'strong bidder' at later auctions. This, in turn, leads to aggressive bidding for the first object, hence an explanation for declining prices over the auction sequence. The current paper allows sellers to respond optimally by designing selling mechanisms that discriminate against strong bidders, i.e. collectors of complementary objects.

More precisely, the current paper analyzes the optimal auction problem in a framework where two complementary objects become available over time. Initially, it is assumed that two independent auction designers device an auction mechanism without taking into account possible adverse affects on rival auctioneers this might have. This corresponds to a situation where the two objects are sold by independent sellers. In the second part of the paper, a situation where auction designers cooperate is analyzed. This might be interpreted as a situation where the different objects are auctioned over time by the same seller. It is argued that there is indeed scope for increasing joint expected revenue since there is an externality between two independent sellers, who will therefore not necessarily choose an auction mechanism that is jointly optimal when acting independently. A comparison of the two scenarios reveals that the cooperative auction design involves lower reservation prices at the first auction and less discrimination against the strong bidder at the late auction, compared to the case of independent (noncooperative) auction designers. In general, none of the two scenarios lead to an efficient allocation of the objects. However, cooperative auction designers tend to allocate the two objects to the same bidder more often than independent auction designers do.

The paper proceeds as follows. Section 2 sets the stage by introducing the optimal auction problem in the standard private values model where only one object is auctioned. Section 3 outlines a model of sequential sales where objects are complements. This model will be the subject of analysis in the remaining sections. The case where auction designers maximize revenue independently is studied in section 4, and section 5 considers the effect of designing a set of joint maximizing mechanisms by cooperating sellers. Finally, section 6 offers some concluding remarks.

2 Preliminaries

The main purpose of this section is to introduce the tools and concepts of mechanism design which are useful when studying optimal selling mechanisms. While the definitions and theorems in this section can be found elsewhere, see e.g. A. Mas-Colell, M. Whinston and J. Green (1995), they are included as an introduction to the notation used throughout the paper. Moreover, the problem of characterizing optimal sequential auction mechanisms for complementary objects can be formulated as an application of the standard optimal auction problem for a single object. Consequently, subsequent sections will refer heavily to the basic insights developed in this section.

2.1 The optimal auction problem

The literature on optimal auctions begins with the work of W. Vickrey (1961). Vickrey considered the following situation: A seller values an item at zero, and he has the option to sell it to one of n risk-neutral bidders. Each bidder alone knows the value that he places on the object. Because the seller and the other buyers are uncertain about this value, it appears to them to be a random variable. Formally, assume each of n bidders, indexed by $i = 1, .., n$, draw a private value, v_i, from a distribution function $F_i(v_i)$ defined on $[\underline{v}_i, \overline{v}_i]$. The associated density function is denoted by $f_i(v_i)$ which is everywhere strictly positive on $[\underline{v}_i, \overline{v}_i]$ for all bidders $i = 1, .., n$. For simplicity, it is assumed that valuations are independent variables.[1] The task is now to device a selling mechanism that yields the highest possible expected revenue to the seller.

The problem of designing an optimal selling mechanism among all con-

[1]This framework is refered to in the literature as the independent private values model.

ceivable mechanisms may seem like a formidable task. However, the key observation is to note that the revelation principle applies. Hence, attention can be restricted to incentive-compatible direct selling mechanisms. In short, the revelation principle states that, if some selling mechanism yields the seller expected revenue equal to R, then so too does some incentive-compatible direct selling mechanism. But this means that no selling mechanism among all conceivable selling mechanisms yields more expected revenue for the seller than the revenue-maximizing incentive-compatible direct selling mechanism. Thus, we can restrict attention to the manageable set of incentive-compatible direct selling mechanisms. In this way, the optimal auction problem has been simplified considerably while losing nothing.[2] A few formal definitions are necessary at this point.

Definition 1 ***A direct selling mechanism*** *is a collection of* n *probability assignment functions,* $p_1(v_1,..,v_n),..,p_n(v_1,..,v_n)$*, and* n *transfer functions,* $t_1(v_1,..,v_n),..,t_n(v_1,..,v_n)$*. For each* i *and every vector of valuations* $(v_1,..,v_n)$*,* $p_i(v_1,..,v_n) \in [0,1]$ *denotes the probability that bidder* i *receives the object and* $t_i(v_1,..,v_n) \in R$ *denotes the payment that bidder* i *must make to the seller. Moreover,* $\sum_{i=1}^{n} p_i(v_1,..,v_n) \leq 1$ *is required.*

A direct selling mechanism works as follows: Because the seller does not know the bidders' values, he asks them to report them to him simultaneously. He then takes those reports, $(\widehat{v}_1,..,\widehat{v}_n)$, which need not be truthful, and assigns the object to one of the bidders according to the probabilities $p_i(\widehat{v}_1,..,\widehat{v}_n)$, $i = 1,..,n$, keeping the object with the residual probability,

[2] Except for the usual caveats about uniqueness of the incentive-compatible direct selling mechanism. The revelation principle implies that the direct revelation game has one equilibrium that yields the same allocation as the original equilibrium. However, this equilibrium need not be unique.

$p_0 = 1 - \sum_{i=1}^{n} p_i$. Finally, the seller secures the payment $t_i(\widehat{v}_1, .., \widehat{v}_n)$ from each bidder $i = 1, .., n$. The distinguishing feature of a direct mechanism is that the message space for each agent is equal to the type-space, or, in this case, the space of valuations.

An incentive compatible direct selling mechanism is a direct selling mechanism in which each bidder must find it optimal to report his true value given that all other bidders do so. In other words, truthtelling is a Bayesian Nash equilibrium in an incentive-compatible direct selling mechanism. To simplify notation, define the 'expected probability assignment function' of bidder i as follows:[3]

$$\widetilde{p}_i(v_i) = \int_{\underline{v}_1}^{\overline{v}_1} .. \int_{\underline{v}_n}^{\overline{v}_n} p_i(v_i, v_{-i}) \cdot f_1(v_1)..f_n(v_n) \cdot dv_1..dv_n \tag{1}$$

and make an analogous definition of bidder i's 'expected transfer function':

$$\widetilde{t}_i(v_i) = \int_{\underline{v}_1}^{\overline{v}_1} .. \int_{\underline{v}_n}^{\overline{v}_n} t_i(v_i, v_{-i}) \cdot f_1(v_1)..f_n(v_n) \cdot dv_1..dv_n \tag{2}$$

An incentive-compatible direct selling mechanism can then be characterized by the following theorem.

Theorem 1 *(A. Mas-Colell, M. Whinston and J. Green (1995)) A direct selling mechanism* $\{p_i(v_1, .., v_n),\ t_i(v_1, .., v_n)\}_{i=1}^{n}$ *is incentive-compatible iff*

(i) $\forall(i, v_i) : \widetilde{p}_i(v_i)$ *is nondecreasing and*

(ii) $\forall(i, v_i) : \widetilde{p}_i(v_i)v_i - \widetilde{t}_i(v_i) = \widetilde{p}_i(\underline{v}_i)\underline{v}_i - \widetilde{t}_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} \widetilde{p}_i(x)dx$

The task of finding a selling mechanism that maximizes the seller's expected revenue has now been reduced to doing a search among all individually

[3]The integrals in (1) and (2) are only over dv_j for $j \neq i$.

rational, incentive-compatible direct selling mechanisms. Hence, the task has been reduced to choosing the functions $p_i(.)$ and $t_i(.)$ to maximize expected revenue, R:

$$\max_{p_i(.),t_i(.)} R = \int_{\underline{v}_1}^{\overline{v}_1} .. \int_{\underline{v}_n}^{\overline{v}_n} \sum_{i=1}^{n} t_i(v_1,..,v_n) \cdot f_1(v_1)..f_n(v_n) \cdot dv_1..dv_n \tag{3}$$

subject to the following conditions:

$$\begin{aligned}
&(Q) \quad \forall(i,v_i) : p_i(v_1,..,v_n) \in [0,1], \text{ and } \forall(v_i) : \sum_{i=1}^{n} p_i(v_1,..,v_n) \leq 1 \\
&(IC_1) \quad \forall(i,v_i) : \widetilde{p}_i(v_i) \text{ is nondecreasing} \\
&(IC_2) \quad \forall(i,v_i) : \widetilde{p}_i(v_i)v_i - \widetilde{t}_i(v_i) = \widetilde{p}_i(\underline{v}_i)\underline{v}_i - \widetilde{t}_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} \widetilde{p}_i(x)dx \\
&(IR) \quad \forall(i,v_i) : \widetilde{p}_i(v_i)v_i - \widetilde{t}_i(v_i) \geq 0
\end{aligned} \tag{4}$$

where (Q) can be interpreted as a quantity constraint, stating that only one unit is to be allocated among the buyers and the seller. (IC_1) and (IC_2) are the two conditions for incentive-compatibility, and (IR) is the condition that no bidder has negative expected utility from participating since participation is voluntary.

To simplify, rearrange the expression for the seller's revenue by inserting (IC_2) into the objective function. After a few manipulations, problem (3) can be restated as

$$\begin{aligned}
\max_{p_i(.),t_i(.)} R = \; & \int_{\underline{v}_1}^{\overline{v}_1} .. \int_{\underline{v}_n}^{\overline{v}_n} \left\{ \sum_{i=1}^{n} p_i(v_1,..,v_n) \cdot \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right] \right\} \cdot f_1(v_1)..f_n(v_n) \cdot dv_1..dv_n \\
& - \sum_{i=1}^{n} \left[\underline{v}_i \widetilde{p}_i(\underline{v}_i) - \widetilde{t}_i(\underline{v}_i) \right]
\end{aligned} \tag{5}$$

subject to (Q), (IC_1) and (IR).

First, it is clear that last term in (5) must be set equal to zero in an optimal selling mechanism. The last term is the sum of all bidders' expected surplus evaluated at the lowest possible valuation. Individual rationality, (IR), requires this term to be non-negative, but there is no reason to leave any money on the table for the lowest valuation type. Hence, an optimal mechanism, $\{p_i^*(.), t_i^*(.)\}_{i=1}^n$, must satisfy

$$\forall(i) : \underline{v}_i \widetilde{p}_i(\underline{v}_i) = \widetilde{t}_i(\underline{v}_i) \tag{6}$$

Attention can now be restricted to the first term in (5). Clearly this is maximized if the term in curly brackets is maximized for each vector of valuations, $(v_1, .., v_n)$. The following probability assignment function does the trick

$$p_i^*(v_1, .., v_n) = \begin{cases} 1 & \text{if } v_i - \frac{1-F_i(v_i)}{f_i(v_i)} > \max\left\{0, v_j - \frac{1-F_j(v_j)}{f_j(v_j)}\right\} \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Hence, the object is allocated to the bidder with the highest $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ if this is positive. To ensure that (IC_1) is satisfied, and (7) is indeed a solution to problem (5), the following regularity condition is needed

$$v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \text{ is strictly increasing in } v_i \tag{8}$$

The final part of the optimal mechanism is a set of transfer functions compatible with condition (6) and the allocation rule (7). This transfer function can be determined directly from (IC_2):

$$\widetilde{t}_i^*(v_i) = \widetilde{p}_i^*(v_i)v_i - \int_{\underline{v}_i}^{v_i} \widetilde{p}_i^*(x)dx \tag{9}$$

Since $\widetilde{t}_i^*(.)$ and $\widetilde{p}_i^*$ are 'averages', (9) holds if the required relationship holds for every vector of valuations. That is,

$$t_i^*(v_1, .., v_n) = p_i^*(v_1, .., v_n) \cdot v_i - \int_{\underline{v}_i}^{v_i} p_i^*(x, v_{-i}) \cdot dx \tag{10}$$

satisfies (9). The solution to the optimal auction problem is summarized in the following theorem:

Theorem 2 *(A. Mas-Colell, M. Whinston and J. Green (1995)) In the private values framework introduced above, assuming that (8) holds, an optimal incentive-compatible direct selling mechanism is characterized by the allocation rules (7) and the transfer functions (10).*

This completes the derivation of the optimal selling mechanism for a single object. As mentioned in the introduction, the insights gained in this section will be valuable in later sections when the optimal auction problem is studied for the case of complementary objects that must be allocated sequentially. It turns out that the more complicated problem of allocating complements sequentially does indeed fit into the framework outlined above. Before proceeding to the model of sequential sales, it is useful to introduce the 'marginal revenue interpretation' of the optimal auction problem.

2.2 Marginal revenues

The optimal auction mechanism derived above can be given a simple "marginal revenue" interpretation due to J. Bulow and J. Roberts (1989) among others. Analogous to monopoly problem $[1 - F_i(v)]$ can be interpreted as the "demand curve" for bidder i. Bidder i's value exceeds any v with probability $[1 - F_i(v)]$, so if a monopolist faced the single bidder, i, and set a take-it-or-leave-it offer of price v, the monopolist would make a sale with probability $[1 - F_i(v)]$, that is, the monopolist's expected quantity of sales would be $q_i(v) = [1 - F_i(v)]$. Just as in standard monopoly theory, the monopolist's revenues can be computed as price times quantity, hence revenue from bidder

i equals $v \cdot q_i(v)$. Using the inverse demand curve, $v_i(q)$ (which is well defined under assumption (8)), marginal revenue from bidder i is given by

$$MR_i(v_i) = \frac{d}{dq}[v_i(q) \cdot q] = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \tag{11}$$

So an auction that always sells to the bidder with the highest marginal revenue, except makes no sale if no bidder's marginal revenue exceeds zero, will maximize expected revenue. Note that the seller does necessarily allocate the object to the bidder with the highest valuation but rather to the bidder with the highest marginal revenue. Furthermore, the seller withholds the object with strictly positive probability since all marginal revenues might be negative. Even though valuations are always positive and the seller values the object at zero, the revenue maximizing selling mechanism is inefficient in the sense that a sale does not always take place. The intuition behind this result is similar to a that of a monopolist who restricts output in order to increase revenue.

The optimal selling mechanism can be implemented as follows: If n bidders have independent private values with bidder i's value drawn from the continuous positive density f_i and each $MR_i(v_i)$ is strictly increasing in v_i, then the following direct selling mechanism yields the seller the largest possible expected revenue: For each reported vector of values, $v = (v_1, .., v_n)$, the seller assigns the object to the bidder i whose $MR_i(v_i)$ is strictly largest and positive. If there is no such bidder, the seller keeps the object and no payments are made. If there is such a bidder i, then only this bidder makes a payment to the seller in the amount r_i^*, where $MR_i(r_i^*) = 0$ or $\max_{j \neq i}\{MR_j(v_j)\}$, whichever is largest. Bidder i's payment, r_i^*, is, therefore, the largest value he could have reported, given the others' reported values, without receiving the object.

It is important to note that the marginal revenue interpretation of the optimal auction problem does not require any further assumptions, apart from assumptions already made. The marginal revenue approach is merely a reformulation of the optimal auction problem into terms that can be given a more conventional economic interpretation. For example, the analogy to revenue maximizing monopoly. With these last remarks in mind, we are now ready to analyze the problem of optimal sequential selling mechanisms for complementary objects. The next section introduces the basic framework.

3 Sequential sales of complementary objects

This section outlines a framework where two complementary objects become available over time. The study of optimal auctions in subsequent sections is restricted to this framework. Let θ_i^k denote bidder i's valuation for object k alone,[4] for all bidders $i = 1, .., n$ and all objects $k = 1, 2$. Assume θ_i^k is drawn from a distribution function $G(.)$ defined on $[\underline{\theta}, \overline{\theta}]$, hence all valuations are independent and identically distributed across bidders and objects. Denote the corresponding density function by $g(.)$, which is strictly positive and differentiable on the support.

The complementarity will be modelled as follows: If bidder i draws a pair of valuations, (θ_i^1, θ_i^2), then the valuation of the bundle is $\theta_i^1 + \theta_i^2 + \alpha$, where $\alpha \geq 0$ is a constant known by all bidders and sellers. Hence, the value of the bundle is larger than the sum of individual valuations. Let $x_i^k = 1$ if bidder i receives object k and $x_i^k = 0$ otherwise.[5] The utility of bidder i can then be represented by the following utility function:

$$u_i(x_i^1, x_i^2) = \theta_i^1 x_i^1 + \theta_i^2 x_i^2 + \alpha x_i^1 x_i^2 - T_i \tag{12}$$

[4] Throughout the paper subscripts denote bidders and superscripts denote objects/auctions.

[5] $x^k = \{x_1^k, .., x_n^k\}$ is thus a realization of the probability assignment functions for each object, $k = 1, 2$.

where T_i denotes bidder i's total payment to the two sellers.

Assume object one and two are allocated by two mechanisms, Γ^1 and Γ^2. For the moment, disregard the timing of announcement of auction design. The sequence of events is then

1. Bidders draw valuations for object one, $\theta^1 = (\theta^1_1, .., \theta^1_n)$.
2. Object one is allocated by mechanism one, Γ^1. The resulting allocation, $x^1 = (x^1_1, .., x^1_n)$, is publicly observable after this stage.
3. Bidders draw valuations for object two, $\theta^2 = (\theta^2_1, .., \theta^2_n)$.
4. Object two is allocated by mechanism two, $\Gamma^2(x^1)$. This mechanism may use the information contained in the allocation of object one.

The analysis of optimal auctions in this framework is particularly tractable for two reasons. First, the fact that valuations are independent across objects and bidders implies that release of information after the first auction is not a design variable for the first seller. Second, the only information, which the second seller can condition the allocation mechanism on, is the allocation of the first object. This is assumed to be public information after the first auction is conducted, hence this is not a design variable for the first auction designer either. The optimal auction problem is thus reduced to determining optimal probability assignment functions and transfer functions as in the case of a single object.

The fact that objects are complements necessitates a distinction between each bidder's *valuation* and his *willingness to pay.* While valuation refers to a bidders utility from receiving a single object alone, willingness to pay refers to the incremental value of receiving an object conditional on the allocation of other objects. In the following, the stochastic variable Θ^k_i denotes bidder i's valuation for object k, and the stochastic variable V^k_i denotes bidder i's willingness to pay for object k. As mentioned above, the distribution of Θ_i^k is denoted by $G(.)$. When reference is made to willingness to pay, V_i^k, the distribution function is denoted by $F_i^k(.)$ and the corresponding density function is denoted by $f_i^k(.)$. The relation between valuation and willingness to pay is discussed next.

It is clear, from the sequential structure outline above, that each bidder's willingness to pay for object two, given object one has already been allocated, can be written as

$$v_i^2 = \begin{cases} \theta_i^2 + \alpha & \text{if bidder } i \text{ received object one} \\ \theta_i^2 & \text{otherwise} \end{cases} \tag{13}$$

Only the bidder who received object one can realize the complementarity. The relation between valuation and willingness to pay for object two is thus given by

$$V_i^2 = \Theta_i^2 + \alpha x_i^1 \tag{14}$$

In the following sections we shall sometimes refer to the winner of object one as the incumbent and the other bidders will be referred to as entrants.

Similarly, the willingness to pay for object one may differ from the valuation for that object alone. However, because of the sequential structure of the game, the willingness to pay for object one cannot be conditioned on the allocation of object two. Rather, it must be based on the expected allocation of object two given the allocation of object one. If bidder i receives object one, he gains immediately θ_i^1 and enters auction two as an incumbent. Hence, the winner of object one gains

$$\theta_i^1 + E_{\theta^2}[s_i^2(\theta^2 \mid x_i^1 = 1)] \tag{15}$$

where $E_{\theta^2}[s_i^2(\theta^2 \mid x^1)]$ denotes bidder i's expected surplus from participating in auction two, given the allocation of object one. Similarly, if bidder i does not receive object one, he enters auction two as an entrant and gains

$$E_{\theta^2}[s_i^2(\theta^2 \mid x_i^1 = 0)] \tag{16}$$

Define the surplus from entering the second auction as an incumbent, relative to entering as an entrant, by

$$\Delta = E_{\theta^2}[u_i^2(\theta^2 \mid x_i^1 = 1)] - E_{\theta^2}[u_i^2(\theta^2 \mid x_i^1 = 0)] \tag{17}$$

Bidder i's willingness to pay for object one is then related to valuations as follows

$$V_i^1 = \Theta_i^1 + \Delta \tag{18}$$

It is immediately clear that Δ depends on the specific details of the selling mechanism devised by the second seller. Since bidders' willingness to pay for object one depends on the relative treatment of incumbents and entrants by the second seller, we have identified a potential source of externality between the to auctions. Consequently, there might be scope for increasing expected revenue by coordinating selling mechanisms. Sections 4 and 5 below highlight the sources of externalities between auctions in more detail. This concludes the modelling of bidders' preferences.

3.1 Efficiency

Before we analyze the optimal auction problem in the framework introduced above, a few remarks about the inherent inefficiency of the problem are worth making. Given the sequential nature of the uncertainty, an efficient allocation can never be guaranteed by any selling mechanism. Since the first object must be allocated to a bidder before valuations for the second object are known, no mechanism can guarantee efficiency. Instead, the best allocation (from an efficiency point of view) is constrained efficient. Clearly, object one should be allocated to the bidder with the highest valuation, but object two should be allocated to the bidder with the highest willingness to pay, i.e. valuation including complementarities. Formally, constrained efficiency can be defined as follows:

Definition 2 *An allocation* (x^1, x^2) *is* ***constrained efficient*** *iff*

$$x_i^1 = \begin{cases} 1 & \text{if } v_i^1 \geq v_j^1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad x_i^2 = \begin{cases} 1 & \text{if } v_i^2 + \alpha x_i^1 \geq v_j^2 + \alpha x_j^1 \\ 0 & \text{otherwise} \end{cases}$$

This definition of efficiency is the proper benchmark for evaluating properties of optimal selling mechanisms in subsequent sections. Also, note that a sequence of second-price auctions (sealed or open) without reserve prices results in the constrained efficient allocation. However, as we will se in the following sections, second-price auctions without reserve prices are not revenue maximizing.

Optimal auctions in the framework outlined in this section can be studied under various assumptions about the objectives of sellers and their ability to cooperate. The question is whether sellers are maximizing joint revenue, by coordinating auction mechanisms, or whether mechanisms are designed independently. Joint maximization of revenue can be interpreted as the situation where all objects are sold by the same seller. This is the subject of analysis in section 5 below. First, section 4 analyses the case of independent maximizers, which can be interpreted as independent sellers without the ability to collude.

4 Independent auction designers

This section analyzes the sequence of optimal auctions designed by independent sellers. The first auctioneer is thus seeking to maximize expected revenue without taking into account any externalities imposed on the second auctioneer. Similarly, the second seller responds to the allocation of object one by devising a selling mechanism that maximizes his own expected revenue. Since the second seller knows the design and outcome of the first auction, at the time of designing auction two, the problem can be solved by backwards induction.

4.1 Auction two

Given the outcome of the first auction, the second seller is faced with n bidders, where one of them might have received object one. Since information about which bidder, if any, received object one is public information, the second seller can condition the optimal auction mechanism on the outcome of auction one. Consider the following two cases in turn: (i) Object one is not sold and (ii) object one is sold to bidder 1 (without loss of generality).

- **Case 1: Object one not sold, $x_i^1 = 0$ for all $i = 1, .., n$**

If object one was kept by the seller, each bidder's willingness to pay for object two is equal to the valuation for object two alone. Formally, $V_i^2 = \Theta_i^2$ for $i = 1, .., n$. Hence, the distribution functions of the two variables are related by $F_i^2(.) = G(.)$, and the problem is reduced to the simple symmetric special case of the analysis in section 2. The optimal allocation rule (7) can thus immediately be reformulated in terms of θ^2:

$$p_i^2(\theta_1^2, .., \theta_n^2) = \begin{cases} 1 & \text{if } MR_i^2(\theta_i^2) > \max\left\{0, MR_j^2(\theta_j^2)\right\} \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

where marginal revenues are given by

$$MR_i^2(\theta_i^2) = \theta_i^2 - \frac{1 - G(\theta_i^2)}{g(\theta_i^2)} \tag{20}$$

One implication is that object two is kept by the second seller with positive probability. To see this, note that optimal reserve level of θ_i^2, r, is determined implicitly by

$$r = \frac{1 - G(r)}{g(r)} \tag{21}$$

Since the reserve prices are identical for all bidders, object two is allocated (constrained) efficiently if it is sold. The only source of inefficiency in this case is the possibility that the seller keeps the object. Naturally, the complete allocation of both objects can never be constrained efficient in this case since it was assumed that object one was not sold. This completes the case where object one was not sold.

- **Case 2: Object one allocated to bidder i, $x_i^1 = 1$ and $x_j^1 = 0$ for all $j \neq i$**

The problem faced by the second seller in this case is also a standard optimal auction problem as studied in section 2 above. However, bidders are asymmetric because one bidder enters auction two as an incumbent. Without loss of generality it can be assumed that the identity of the incumbent bidder is bidder 1. The relation between valuations and willingness to pay is thus given by

$$V_i^2 = \Theta_i^2 + \alpha x_i^1 \tag{22}$$

Note that the distribution of willingness to pay is related to the distribution of valuation in the following way

$$F_1^2(v_1^2) = p(V_1^2 \leq v_1^2) = p(\Theta_1^2 + \alpha \leq v_1^2) = G(v_1^2 - \alpha) \tag{23}$$

and

$$F_i^2(v_i^2) = p(V_i^2 \leq v_i^2) = p(\Theta_i^2 \leq v_i^2) = G(v_i^2) \tag{24}$$

Hence, the optimal allocation rule can be specified as follows:[6]

$$p_i^2(\theta_1^2, .., \theta_n^2) = \begin{cases} 1 & \text{if } MR_i^2(\theta_i^2) > \max\left\{0, MR_j^2(\theta_j^2)\right\} \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

where the marginal revenue for bidder 1 is

$$MR_1^2(\theta_1^2) = \theta_1^2 + \alpha - \frac{1 - G(\theta_1^2)}{g(\theta_1^2)} \tag{26}$$

and for all other bidders

$$MR_i^2(\theta_i^2) = \theta_i^2 - \frac{1 - G(\theta_i^2)}{g(\theta_i^2)} \qquad \text{for } i = 2, .., n \tag{27}$$

The first thing to note about this allocation rule is that the seller keeps object two with positive probability, as in case 1 above. Furthermore, reserve prices are asymmetric. Denote the reserve level of θ_i^2 for each bidder by r_i, then relative reserve prices for the two types of bidders can be stated in the following proposition.

Proposition 1 $r_1 < r_2 = .. = r_n$

Proof. r_1 is implicitly defined by $r_1 - \frac{1-G(r_1)}{g(r_1)} = -\alpha$ and r_i is implicitly defined by $r_i - \frac{1-G(r_i)}{g(r_i)} = 0$ for bidders $i = 2, .., n$. Since marginal revenues are assumed to be strictly increasing in valuations the result follows immediately. ■

Proposition 1 implies that the incumbent bidder can receive object two without having the largest valuation for object two alone. However, the interesting question is whether the optimal mechanism favors the incumbent

[6]Note, that the relation between $F_i^2(.)$ and $G(.)$ is common knowledge. Therefore, the seller can either ask bidders to report their 'valuation' or their 'willingness to pay'. Either way, one can be computed from the other.

bidder more or less than constrained efficiency requires. Recall that constrained efficiency allocates object two to bidder 1 if $\theta_1^2+\alpha > \max\{\theta_2^2,..,\theta_n^2\}$. It turns out that the optimal selling mechanism for object two may either bundle objects too much or too little (relative to constrained efficiency), depending on the distribution of valuations and the size of the complementarity. To see this, consider the following three cases.

Case 1 *Too much bundling*
If $\theta_1^2 < \theta_1^2 + \alpha < \max\{\theta_2^2,..,\theta_n^2\}$ *some bidder* $j \neq 1$ *should receive object two according to constrained efficiency. However, bidder* 1 *receives object two if* $\theta_1^2 + \alpha - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} > \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\}$. *Since we only know that* $\theta_1^2 - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} < \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\}$, *the optimal selling mechanism may bundle too much.*

Case 2 *Too little bundling*
If $\theta_1^2 < \max_{j=2,..,n}\{\theta_2^2,..,\theta_n^2\} < \theta_1^2+\alpha$ *bidder* 1 *should receive object two according to constrained efficiency. However, bidder some bidder* $j \neq 1$ *receives object two if* $\theta_1^2 + \alpha - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} < \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\}$. *Since we only know that* $\theta_1^2 - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} < \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\}$ *the optimal mechanism may bundle too little.*

Case 3 *Constrained efficiency*
If $\max_{j=2,..,n}\{\theta_2^2,..,\theta_n^2\} < \theta_1^2 < \theta_1^2+\alpha$ *bidder* 1 *should receive object two according to constrained efficiency. Since the optimal mechanism allocates object two to bidder* 1 *if* $\theta_1^2 + \alpha - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} > \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\}$ *and we know that* $\theta_1^2 - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} > \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\}$ *there is no source of inefficiency in this case.*

The results are summarized in the following proposition.

Proposition 2 *If* $\frac{dMR_i(\theta_i^2)}{d\theta_i^2} = 1$ *the optimal selling mechanism for object two is constrained efficient. If* $\frac{dMR_i(\theta_i^2)}{d\theta_i^2} > 1$ *too little bundling may occur and if* $\frac{dMR_i(\theta_i^2)}{d\theta_i^2} < 1$ *too much bundling may occur.*

Proof. case 1 cannot occur if $\max_{j=2,..,n}\{\theta_2^2,..,\theta_n^2\} - \theta_1^2 > \alpha \Rightarrow \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\} - \theta_1^2 - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} > \alpha$ which is equivalent to $\frac{dMR_i(\theta_i^2)}{d\theta_i^2} > 1$. Similarly, case 2 cannot occur if $\max_{j=2,..,n}\{\theta_2^2,..,\theta_n^2\} - \theta_1^2 < \alpha \Rightarrow \max_{j=2,..,n}\{\theta_j^2 - \frac{1-G(\theta_j^2)}{g(\theta_j^2)}\} - \theta_1^2 - \frac{1-G(\theta_1^2)}{g(\theta_1^2)} < \alpha$ which is equivalent to $\frac{dMR_i(\theta_i^2)}{d\theta_i^2} < 1$. ■

The intuition behind this result is that constrained efficiency requires that object two is allocated to the incumbent bidder if he has a valuation that is at least α higher than the highest valuation of the entrants. While the optimal selling mechanism favors the incumbent, the difference in reserve prices may be higher than α (too much bundling) or less than α (too little bundling). Finally, note that case 1 is less likely to occur if α is large relative to the support of θ_i^2. For large complementarities, too little bundling is therefore the only concern.

Before proceeding to auction one, a few observations about the relative surplus of different types of bidders are worth making. Denote bidder i's surplus form participating in the second auction by $s_i^2(\theta^2, x^1)$. The surplus depends on valuations for object two and the outcome of the first auction. Bidder i can either be an incumbent, an entrant competing with one incumbent or an entrant competing with other entrants only. For simplicity, denote the surplus of bidder i in each of these three cases by $s_i^2(\theta^2, I)$, $s_i^2(\theta^2, E)$ and $s_i^2(\theta^2, \emptyset)$. Also, recall that the surplus to bidder i is completely determined by the probability assignment function

$$s_i(\theta_1, .., \theta_n) = \int_{\underline{\theta}}^{\theta_i} p_i(x, \theta_{-i}) \cdot dx \tag{28}$$

Since the preceding analysis reveals that allocation rules differ between incumbents and entrants, we may write

$$p_i^2(\theta_1^2, .., \theta_n^2) = \begin{cases} p_i^2(\theta_1^2, .., \theta_n^2 \mid I) & \text{if bidder } i \text{ is incumbent} \\ p_i^2(\theta_1^2, .., \theta_n^2 \mid \emptyset) & \text{if no bidder is incumbent} \\ p_i^2(\theta_1^2, .., \theta_n^2 \mid E) & \text{if bidder } i \text{ is entrant} \end{cases} \tag{29}$$

From (19) and (25) it is clear that for all $\theta_1^2, .., \theta_n^2$,

$$p_i^2(\theta_1^2, .., \theta_n^2 \mid I) \geq p_i^2(\theta_1^2, .., \theta_n^2 \mid \emptyset) \geq p_i^2(\theta_1^2, .., \theta_n^2 \mid E) \tag{30}$$

Hence,

$$E_{\theta^2}[s_i^2(\theta^2, I)] > E_{\theta^2}[s_i^2(\theta^2, \emptyset)] > E_{\theta^2}[s_i^2(\theta^2, E)] \tag{31}$$

The expected value from participating in auction two as an incumbent is thus larger than that of an entrant. Also, an entrant prefers to compete against entrants alone instead of facing an incumbent among the bidders. This result will be used below when the optimal selling mechanism for object one is derived.

4.2 Auction one

At this stage, the seller of object one must design an optimal auction taking into account the fact that the seller of object two responds to the allocation of object one as described above. As discussed in section 3, each bidder's willingness to pay is given by his valuation, θ_i^1, plus the expected benefit from entering auction two as an incumbent, relative to entering auction two as an entrant, denoted by Δ. Since a bidder's expected value of participating in auction two depends on the types of competitors he face, Δ depends on the probability that seller one keeps object one. $\Delta(p_0^1)$ can therefore be written as follows

$$\Delta(p_0^1) = E_{\theta^2}[s_i^2(\theta^2, I)] - [p_0^1 E_{\theta^2}[s_i^2(\theta^2, \emptyset)] + (1 - p_0^1) E_{\theta^2}[s_i^2(\theta^2, E)]] \qquad (32)$$

From (31) it can immediately be verified that $\Delta(p_0^1)$ is decreasing in p_0^1. The more likely it is that the seller of object one keeps the object, the less benefit from entering auction two as an incumbent. The relation between the stochastic variables denoting willingness to pay, V_i^1, and valuation, Θ_i^1, is then given by

$$V_i^1 = \Theta_i^1 + \Delta(p_0^1) \qquad \text{for } i = 1, .., n \qquad (33)$$

Marginal revenues can therefore be expressed in terms of valuations by the following expression:

$$MR_i^1(\theta_i^1, p_0^1) = \theta_i^1 + \Delta(p_0^1) - \frac{1 - G(\theta_i^1)}{g(\theta_i^1)} \qquad i = 1, .., n \qquad (34)$$

The optimal auction mechanism for object one is then characterized by Theorem 2 using marginal revenues from (34). Despite the complexity caused by p_0^1 in the expression for marginal revenue, this approach is valid since all the assumptions made in section 2 are satisfied. In particular, we only need to make sure that the revelation principle applies and that marginal revenues are strictly increasing in valuations. Recall, that the revelation principle shows that outcomes resulting from *any* equilibrium of *any* mechanism can be replicated by a truthful equilibrium of some direct mechanism. Since the seller is restricted to choosing a mechanism (as defined in section 2), the revelation principle does indeed apply in this case. Moreover, in order to assure incentive compatibility of the optimal mechanism, we must verify that marginal revenues defined in (34) are strictly increasing in valuations. This is in fact the case since

$$\theta_i^1 \uparrow \quad \Rightarrow \quad p_0^1 \downarrow \quad \Rightarrow \quad \Delta(p_0^1) \uparrow \tag{35}$$

Increasing θ_i^1 implies that the probability that the seller keeps the object is reduced. This in turn implies that $\Delta(p_0^1)$ is increased, which can be seen from (32). Since Δ is increasing in θ_i^1 and the rest of the marginal revenue expression (34) is assumed to be increasing in θ_i^1 the result follows.

At this point it has only been argued that the optimal auction mechanism for the seller of object one is characterized by Theorem 2 using marginal revenues from (34). This observation does not provide too much insight into the nature of the optimal auction mechanism. However, it can immediately be established that the optimal auction is nondiscriminatory. Assume, for the moment, that p_0^1 is fixed at some level, which implies that Δ is constant. The optimal allocation rule then takes the following form

$$p_i^1(\theta_1^1, .., \theta_n^1) = \begin{cases} 1 & \text{if } MR_i^1(\theta_i^1) > \max\left\{0, MR_j^1(\theta_j^1)\right\} \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

Hence, object one is allocated to the bidder with the largest marginal revenue, provided this is positive, as in the simple case of section 2. However, the auction designer has an additional choice variable in p_0^1. Since Δ varies with p_0^1 the seller determines the optimal p_0^1 by taking into account the effect on marginal revenues. For example, lowering the 'cut-off level' of MR_i^1 from 0 to some negative number, k, each bidder's marginal revenue increases. To see this, note that p_0^1 is implicitly determined by k in the following expression

$$p_0^1 = prob(\max_{i=1,..,n} \{MR_i^1(\theta_i^1, p_0^1)\} < k) \tag{37}$$

and note that

$$\frac{dp_0^1}{dk} = -\frac{\frac{\partial}{\partial k}(prob(\max_{i=1,..,n} \{MR_i^1(\theta_i^1, p_0^1)\} < k))}{\frac{\partial}{\partial p_0}(prob(\max_{i=1,..,n} \{MR_i(\theta_i^1, p_0^1)\} < k)) - 1} > 0 \tag{38}$$

Hence, $p_0^1(k)$ is increasing in k (as when marginal revenues are constant). The optimal auction problem has now been reduced to determining k in the following allocation rule

$$p_i^1(\theta_1^1, .., \theta_n^1) = \begin{cases} 1 & \text{if } MR_i^1(\theta_i^1, p_0^1(k)) > \max\left\{k, MR_j^1(\theta_j^1, p_0^1(k))\right\} \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

First, note that $k > 0$ can never be optimal. If the cut-off level is increased from zero to some positive number, marginal revenues decrease and the probability of sale is reduced. Consider $k < 0$, hence $MR_i^1(\theta_i^1, p_0^1(k)) > MR_i^1(\theta_i^1, p_0^1(0))$ for all θ_i^1 and $i = 1, .., n$. The change in revenue can be written as follows:

$$R(\theta^1, k) - R(\theta^1, 0) = \max_{i=1,..,n} \{MR_i^1(k), k\} - \max_{i=1,..,n} \{MR_i^1(0), 0\} \tag{40}$$

Where $MR_i^1(\theta_i^1, p_0^1(k))$ is denoted by $MR_i^1(k)$ for simplicity. To determine whether this term is positive or negative, rewrite (40) as follows

$$R(\theta^1, k) - R(\theta^1, 0) =$$

$$\begin{cases} \max_{i=1,..,n} \{MR_i^1(k) - MR_i^1(0)\} & \text{if } k < 0 < \max_{i=1,..,n} \{MR_i^1(0)\} < \max_{i=1,..,n} \{MR_i^1(k)\} \\ \max_{i=1,..,n} \{MR_i^1(k)\} & \text{if } k < \max_{i=1,..,n} \{MR_i^1(0)\} < 0 < \max_{i=1,..,n} \{MR_i^1(k)\} \\ \max_{i=1,..,n} \{MR_i^1(k)\} & \text{if } \max_{i=1,..,n} \{MR_i^1(0)\} < k < 0 < \max_{i=1,..,n} \{MR_i^1(k)\} \\ \max_{i=1,..,n} \{MR_i^1(k)\} & \text{if } \max_{i=1,..,n} \{MR_i^1(0)\} < k < \max_{i=1,..,n} \{MR_i^1(k)\} < 0 \\ 0 & \text{if } \max_{i=1,..,n} \{MR_i^1(0)\} < \max_{i=1,..,n} \{MR_i^1(k)\} < k < 0 \end{cases} \tag{41}$$

The first three terms are positive since $\max_{i=1,..,n}\{MR_i^1(k)\}$ is positive. In these cases the effect of lowering the reserve level of MR from 0 to $k<0$ is strictly positive since the object is never sold to a bidder with negative marginal revenue and marginal revenues increase as the reserve level of MR is decreased. However, the fourth term is negative since $\max_{i=1,..,n}\{MR_i^1(k)\}$ is negative. In this case object one is allocated to a bidder with negative marginal revenue. In general, the expected value of decreasing the reserve level of MR depends on the probability weights on the five terms in (41). The only thing that can be concluded is that the optimal reserve level of MR is less than or equal to zero.

The optimal selling mechanism for object one is thus non-discriminatory and the seller sets a reserve level of MR below zero since bidders' willingness to pay depend on the auction design itself. One interpretation of the solution to the optimal auction problem is that the seller equates marginal revenue with marginal cost. Usually, the marginal cost of allocating an object to a given bidder is zero. However, in this setup, the marginal cost is negative since bidders' willingness to pay actually increase if the probability of sale increase. Hence, k can be interpreted as the seller's marginal cost analogous to bidders' marginal revenues.

To sum up, the sequence of optimal selling mechanisms by independent sellers can be characterized as follows. The first seller device a non-discriminatory auction with a reserve price that is less than the reserve price in a one-shot auction. If object one is kept by the seller, the second seller device a nondiscriminatory auction with standard reserve prices. If object one is allocated to a bidder, the second seller device a discriminatory auction that favors the incumbent in the sense that the incumbent receives object two for a lower realized valuation than other bidders. Moreover, the second auction may bundle too much or too little compared to the constrained efficient allocation, which could be obtained by implementing two second-price auctions without reserve prices.

5 Coordinated auction design

This section considers the optimal auction problem when sellers are allowed to coordinate auction mechanisms in order to maximize joint revenue. To keep the focus on sequential sales, attention is restricted to *sequential mechanisms* in the sense that each object is allocated by a standard mechanism where bidders submit one message/bid. This specification rules out a grand mechanism where both objects are allocated simultaneously and mechanisms where the allocation rules depend on bids submitted to earlier mechanisms as well as bids submitted to the current mechanism.

More precisely, a sequential mechanism, in the present framework with two stochastically independent objects, consists of two sub-mechanisms, $\{\Gamma^1, \Gamma^2\}$, where each sub-mechanism, Γ^k, consists of a message space for each agent, $\{M_1^k, .., M_n^k\}$, and an allocation rule,[7] y_m^k, for $k = 1, 2$. In the first sub-mechanism, the allocation rule can only depend on the messages sent, $y_m^1(M^1)$, but the allocation rule specified by the second sub-mechanism may depend on the actual allocation resulting from the first sub-mechanism as well as the messages sent to the second sub-mechanism, i.e. $y_m^2(M^2, y^1)$. Formally, a sequential mechanism is defined as follows:

Definition 3 *A* ***sequential mechanism*** *is a set of sub-mechanisms,* $\Gamma = \{\Gamma^1, \Gamma^2\}$*, where* $\Gamma^1 = \{M^1, y_m^1(M^1)\}$ *and* $\Gamma^2 = \{M^2, y_m^2\left(M^2, y_m^1\right)\}$*.*

[7] In the optimal auction framework, an allocation rule consists of a set of probability assignment functions and a set of transfer functions. Hence, $y = \{p_1, .., p_n, t_1, .., t_n\}$ using the notation of the previous sections.

The problem faced by the sellers is now to device an optimal sequential mechanism chosen among all possible sequential mechanisms. Fortunately, the revelation principle extends to sequential mechanisms as defined above.

Theorem 3 *The revelation principle for sequential mechanisms: Suppose a sequential mechanism with message spaces* $\{M^1,\ M^2\}$ *and allocation functions* $y_m^1(M^1)$ *and* $y_m^2\,(M^2, y_m^1)$ *has a Bayesian Nash equilibrium, then there exists a sequential-mechanism consisting of incentive compatible direct sub-mechanisms that results in the same allocation.*

Proof. See the appendix. ■

With this result in mind, we can now proceed along the same lines as in previous sections. Assume that both sub-mechanisms are announced at the outset, and that this announcement is credible.[8] As in section 5 the relation between willingness to pay and valuation for each bidder is given by

$$V_i^1 = \Theta_i^1 + \Delta(\Gamma^1, \Gamma^2) \qquad i = 1, .., n \tag{42}$$

and

$$V_i^2 = \Theta_i^2 + \alpha x_i^1 \qquad i = 1, .., n \tag{43}$$

As argued above, Δ may depend on the details of both sub-mechanisms. Δ depends on the first sub-mechanism because take into account the probability of facing an incumbent in auction two. The dependence of Δ on the second sub-mechanism is due to the fact that bidders value the prospect of being an incumbent in the second auction. The value of being an incumbent, relative

[8] In general, there are 4 possible cases to consider: Coordinated or independent revenue maximization both with or without commitment. In this paper we restrict attention to cases. Independent sellers without commitment (section 5) and joint revenue maximizing sellers with commitment (section 6).

to being an entrant, naturally depends on the details of the second sub-mechanism, i.e. how the mechanism discriminates between an incumbent and entrants. As in section 5, it must be verified that the marginal revenue approach applies, or that the revelation principle applies and marginal revenues are strictly increasing in valuations. The appendix shows that a sequential revelation principle applies to the problem and marginal revenues are strictly increasing by an argument similar to that of section 5.

The joint optimal selling mechanism (set of sub-mechanisms) can now be derived. Using the sequential revelation principle stated in theorem 3, and going through the same steps as in section 2 leads to the following maximization problem

$$\max_{p_i^k(.),t_i^k(.)} R^1 + R^2 =$$

$$\int_{\underline{v}_1^1}^{\overline{v}_1^1} .. \int_{\underline{v}_n^1}^{\overline{v}_n^1} \left\{ \sum_{i=1}^{n} p_i^1(v_1^1, .., v_n^1) \cdot \left[v_i^1 - \tfrac{1-F_i^1(v_i^1)}{f_i^1(v_i^1)} \right] \right\} \cdot f_1^1(v_1^1)..f_n^1(v_n^1) \cdot dv_1^1..dv_n^1$$

$$+ \int_{\underline{v}_1^2}^{\overline{v}_1^2} .. \int_{\underline{v}_n^2}^{\overline{v}_n^2} \left\{ \sum_{i=1}^{n} p_i^2(v_1^2, .., v_n^2) \cdot \left[v_i^2 - \tfrac{1-F_i^2(v_i^2)}{f_i^2(v_i^2)} \right] \right\} \cdot f_1^2(v_1^2)..f_n^2(v_n^2) \cdot dv_1^2..dv_n^2$$

$$- \sum_{i=1}^{n} \left[\underline{v}_i^1 \widetilde{p}_i^1(\underline{v}_i^1) - \widetilde{t}_i^1(\underline{v}_i^1) \right] - \sum_{i=1}^{n} \left[\underline{v}_i^2 \widetilde{p}_i^2(\underline{v}_i^2) - \widetilde{t}_i^2(\underline{v}_i^2) \right] \tag{44}$$

subject to the following conditions

$$(Q) \quad \forall (k, i, v_i^k) : p_i^k(v_1^k, .., v_n^k) \in [0, 1], \text{ and } \forall (k, v_i^k) : \sum_{i=1}^{n} p_i^k(v_1^k, .., v_n^k) \leq 1$$

$$(IC_1) \quad \forall (k, i, v_i^k) : \widetilde{p}_i^k(v_i^k) \text{ is nondecreasing}$$

$$(IR) \quad \forall (k, i, v_i^k) : \widetilde{p}_i^k(\underline{v}_i^k)\underline{v}_i^k - \widetilde{t}_i^k(\underline{v}_i^k) \geq 0 \tag{45}$$

It can immediately be verified that the participation constraint for the first sub-mechanism must be binding, i.e. $\underline{v}_i^1 \widetilde{p}_i^1(\underline{v}_i^1) = \widetilde{t}_i^1(\underline{v}_i^1)$ for all bidders $i = 1, .., n$. However, it is not immediately clear that this is also the case for the second participation constraint. Since v_i^1 is increasing in the surplus left to an incumbent, relative to an entrant (via Δ), it might be optimal to leave surplus for the lowest type incumbent. Suppose bidder i is an incumbent at the second auction and $\widetilde{p}_i^2(\underline{v}_i^2)\underline{v}_i^2 - \widetilde{t}_i^2(\underline{v}_i^2) = k \geq 0$ then the expected cost of doing this is k. The benefit of leaving k to this type is an increase of k in all bidders' valuation for object one, or

$$k \cdot \int_{\underline{v}_1^1}^{\overline{v}_1^1} .. \int_{\underline{v}_n^1}^{\overline{v}_n^1} \left\{ \sum_{i=1}^{n} p_i^1(v_1^1, .., v_n^1) \right\} \cdot f_1^1(v_1^1)..f_n^1(v_n^1) \cdot dv_1^1..dv_n^1 \tag{46}$$

Since the term in curly brackets is less than or equal to one for all valuations, the benefit from leaving k to the lowest type incumbent is less than or equal to k. Only if object one is sold with probability one is the cost of leaving k to the lowest type incumbent recovered. Therefore, we conclude that the participation constraint for the second sub-mechanism must be binding for all types and the last two terms in (44) can be disregarded.

Since the maximization problem (44) looks like the sum of independent sellers' problems from section four, one candidate for an optimal selling mechanism is the probability assignment functions of section four. However, there are some externalities between the two mechanisms that must be taken into account.

Proposition 3 *The selling mechanism for object one defined by (39) imposes an externality on the second seller. More precisely,* $(v_1^2, .., v_n^2)$ *depends on* Γ^1 *through* p_0^1.

Proof. Assume bidder 1 is the incumbent bidder at the second auction if object one was sold. Bidder 1's willingness to pay for object two is thus given by $v_1^2 = \theta_1^2 + \alpha p_0^1$ since bidder 1 is an incumbent with probability p_0^1. This is clearly increasing in p_0^1. ■

This result implies that the joint optimal sequential auction mechanism is characterized by a lower reserve price for object one, compared to the case of independent sellers. Without the externality identified in proposition 3, the optimal sub-mechanism for object one would be identical to the selling mechanism derived in section 5 for independent sellers.

Proposition 4 *The selling mechanism for object two defined by (25) imposes an externality on the first seller. More precisely,* $(v_1^1, .., v_n^1)$ *depends on* Γ^2*.*

Proof. The willingness to pay for object one is given by $v_i^1 = \theta_i^1 + \Delta$ for all bidders $i = 1, .., n$. But since Δ measures the expected benefit from entering auction two as an incumbent, Δ depends on the details of the second selling mechanism. Recall that $\Delta = E_{\theta^2}[s_i^2(I)] - [p_0 E_{\theta^2}[s_i^2(\emptyset)] + (1 - p_0) E_{\theta^2}[s_i^2(E)]]$. Hence, v_i^1 is increasing in $E_{\theta^2}[s_i^2(I)]$ and decreasing in $E_{\theta^2}[s_i^2(\emptyset)]$ and $E_{\theta^2}[s_i^2(E)]$. ■

This result implies that the joint optimal the joint optimal sequential auction mechanism is characterized by a higher $E_{\theta^2}[s_i^2(I)]$ and/or lower $E_{\theta^2}[s_i^2(\emptyset)]$ and $E_{\theta^2}[s_i^2(E)]$, compared to the case of independent sellers. In other words, the optimal sub-mechanism for object two specifies more favorable terms for an incumbent at the second auction since this increases all bidders' willingness to pay for object one. For independent sellers there is a trade-off between not selling the object and increasing revenue when setting optimal reserve prices. This trade-off is altered when joint revenue is maximized since the cost of increasing the reserve price of an incumbent at the second auction affects all bidders' willingness to pay for object one.

6 Concluding remarks

This paper has addressed the question of how sequential sales of complementary objects should be arranged in order to maximize expected revenue. With independent sellers, each maximizing own expected revenue, the optimal sequence of auctions are characterized by a symmetric first auction followed by an asymmetric second auction. However, independent sellers do not take into account the externalities imposed on other sellers. These externalities arise because the willingness to pay at both auctions are determined by the auction design itself. Therefore, a coordinated auction design with less discrimination against the incumbent bidder at the second auction will increase joint expected revenue. Furthermore, the incentive for the first seller to withhold the object is reduced under joint revenue maximization.

It is clear that the analysis carried out in this paper does not provide a complete characterization of optimal sequential auctions for complementary objects. The level of generality implies a rather vague characterization of optimal selling mechanisms. Clearly, further research on this subject is warranted. Furthermore, it is not clear to what extent the differences between sections 5 and 6 are due to lack of commitment in the independent sellers case, or due to the externalities. The main contribution of this paper is thus methodological rather than specific characterizations of optimal selling mechanisms For example, propositions 3 and 4 provide some general insights about the existence of externalities between sellers. A simplifying example could provide further insights that may be valuable.

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Appendix

This appendix provides a proof of the sequential revelation principle stated in theorem 3. The structure of the proof is roughly the following:[9] Given any sub-mechanism for allocating the first object, the revelation principle applies to the second sub-mechanism. Next, given truthtelling in the second sub-mechanism, the revelation principle applies to the first sub-mechanism.

Assume there are n agents with independent types $\theta = \{\theta_1^1, .., \theta_n^1, \theta_1^2, .., \theta_n^2\}$ in $\times_{i=1}^n \{\Theta_i^1 \times \Theta_i^2\}$. Denote an allocation by $y = \{y^1, y^2\}$ where $y^k \in Y^k$ for $k = 1, 2$, and let the utility of each agent be additively separable:

$$u_i(y, \theta_i) = u_i^1(y^1, \theta_i^1) + u_i^2(y^1, y^2, \theta_i^2) \tag{47}$$

Consider any sequential mechanism, $\Gamma_\mu = \{\Gamma_\mu^1, \Gamma_\mu^2\}$, where messages sent to the two sub-mechanisms are denoted by $\mu^1 = \{\mu_1^1, .., \mu_n^1\}$ and $\mu^2 = \{\mu_1^2, .., \mu_n^2\}$. Consider a Bayesian Nash equilibrium, $\mu_i^*(\theta_i) = \left\{\mu_i^{1*}(\theta_i^1), \mu_i^{2*}(\theta_i^2, y^1)\right\}$, hence by definition, μ_i^{2*} is a best response to μ_{-i}^{2*} given y^{1*} was reached:

$$\begin{aligned} &E_{\theta_{-i}^2}[u_i^2(y^{1*}, y_m^2(\mu_1^{2*}(\theta_1^2, y^{1*}), .., \mu_i^{2*}, .., \mu_n^{2*}(\theta_n^2, y^{1*})), \theta_i^2) \mid \theta_i^2, y^{1*}] \geq \\ &E_{\theta_{-i}^2}[u_i^2(y^{1*}, y_m^2(\mu_1^{2*}(\theta_1^2, y^{1*}), .., \mu_i^2, .., \mu_n^{2*}(\theta_n^2, y^{1*})), \theta_i^2) \mid \theta_i^2, y^{1*}] \end{aligned} \tag{48}$$

[9] Throughout the proof x denotes a vector of variables, $x = \{x_1^1, .., x_n^1, x_1^2, .., x_n^2\}$. Furthermore, x can be partitioned into vectors as follows: $x = \{x^1, x^2\}$ or $x = \{x_1, x_n\}$, where $x^1 = \{x_1^1, .., x_n^1\}$ etc.

and μ_i^{1*} is an initial best response:

$$E_{\theta^1_{-i},\theta^2}[u_i^1(y_m^1(\mu_1^{1*}(\theta_1^1),..,\mu_i^{1*},..,\mu_n^{1*}(\theta_n^1)),\theta_i^1)+$$

$$u_i^2(y_m^1(\mu_1^{1*}(\theta_1^1),..,\mu_i^{1*},..,\mu_n^{1*}(\theta_n^1)),y_m^2(\mu^{2*}(\theta^2,y_m^1(\mu_1^{1*}(\theta_1^1),..,\mu_i^{1*},..,\mu_n^{1*}(\theta_n^1)))),\theta_i^2)\mid\theta_i^1]\geq$$

$$E_{\theta^1_{-i},\theta^2}[u_i^1(y_m^1(\mu_1^{1*}(\theta_1^1),..,\mu_i^{1},..,\mu_n^{1*}(\theta_n^1)),\theta_i^1)+$$

$$u_i^2(y_m^1(\mu_1^{1*}(\theta_1^1),..,\mu_i^{1},..,\mu_n^{1*}(\theta_n^1)),y_m^2(\mu^{2*}(\theta^2,y_m^1(\mu_1^{1*}(\theta_1^1),..,\mu_i^{1},..,\mu_n^{1*}(\theta_n^1)))),\theta_i^2)\mid\theta_i^1] \quad (49)$$

The rest of the proof proceeds in three steps: Step 1 shows that the second sub-mechanism can be replaced with a incentive compatible direct mechanism without changing the equilibrium allocation, y^{2*}, given any y^1. Step 2 shows that the full equilibrium allocation, $\{y^{1*},y^{2*}\}$, can be reached by changing the second sub-mechanism to an incentive compatible direct mechanism. Finally, step 3 shows that the full equilibrium allocation, $\{y^{1*},y^{2*}\}$, can be implemented by changing the first sub-mechanism to an incentive compatible direct mechanism, given that the second sub-mechanism is also an incentive compatible direct mechanism.

- **Step 1: Γ_μ^2 can be replaced by Γ_θ^2.**

Given an allocation y^1, consider any sub-mechanism, Γ_μ^2, with corresponding Bayesian Nash equilibrium denoted by $\{\mu_i^{2*}(\theta_i^2,y^1)\}$. By definition of Bayesian Nash equilibrium,

$$\begin{aligned}&E_{\theta^2_{-i}}[u_i^2(y^1,y_m^2(\mu_1^{2*}(\theta_1^2,y^1),..,\mu_i^{2*},..,\mu_n^{2*}(\theta_n^2,y^1)),\theta_i^2)\mid\theta_i^2,y^1]\geq\\&E_{\theta^2_{-i}}[u_i^2(y^1,y_m^2(\mu_1^{2*}(\theta_1^2,y^1),..,\mu_i^{2},..,\mu_n^{2*}(\theta_n^2,y^1)),\theta_i^2)\mid\theta_i^2,y^1]\end{aligned} \quad (50)$$

Consider a new message space Θ_i^2 for each agent i, so that each agent announces a type $\widehat{\theta}_i^2$ which may differ from the true value θ_i^2. Also, define a new allocation rule, $\overline{y}^2 : \Theta_i^2 \times Y^1 \to Y^2$ by

$$\overline{y}^2(\widehat{\theta}^2, y^1) = y_m^2(\mu^{2*}(\widehat{\theta}^2, y^1)) \tag{51}$$

Denote this new mechanism by $\Gamma_\theta^2 = \{\Theta^2, \overline{y}^2\}$. Since $\{\mu_i^{2*}(\theta_i^2, y^1)\}$ is a Bayesian Nash equilibrium of the original game, defined by Γ_μ^2, it follows immediately that $\{\widehat{\theta}_i^2 = \theta_i^2\}$ is an equilibrium of the game defined by Γ_θ^2:

$$E_{\theta^2_{-i}}[u_i^2(y^1, \overline{y}^2(\theta^2, y^1), \theta_i^2) \mid \theta_i^2, y^1] =$$

$$E_{\theta^2_{-i}}[u_i^2(y^1, y_m^2(\mu^{2*}(\theta^2, y^1)), \theta_i^2) \mid \theta_i^2, y^1] =$$

$$\sup_{\mu_i^2 \in M_i^2} E_{\theta^2_{-i}}[u_i^2(y^1, y_m^2(\mu_1^{2*}(\theta_1^2, y^1), .., \mu_i^2, .., \mu_n^{2*}(\theta_n^2, y^1)), \theta_i^2) \mid \theta_i^2, y^1]$$

$$\geq \sup_{\widehat{\theta}_i^2 \in \Theta_i^2} E_{\theta^2_{-i}}[u_i^2(y^1, \overline{y}^2(\theta_1^2, .., \widehat{\theta}_i^2, .., \theta_n^2, y^1)), \theta_i^2) \mid \theta_i^2, y^1]$$

The first equality results from the definition of the direct revelation mechanism, Γ_θ^2, and the second equality follows from the equilibrium condition, (48). Finally, the weak inequality express the fact that the message spaces of Γ_θ^2 are subsets of the message spaces of Γ_μ^2.

- **Step 2: $\{\Gamma_\mu^1, \Gamma_\mu^2\}$ can be replaced by $\{\Gamma_\mu^1, \Gamma_\theta^2\}$.**

Consider any sequential mechanism, $\Gamma_\mu = \{\Gamma_\mu^1, \Gamma_\mu^2\}$, with corresponding Bayesian Nash equilibrium denoted by $\mu_i^*(\theta_i) = \left\{\mu_i^{1*}(\theta_i^1), \mu_i^{2*}(\theta_i^2, y^{1*})\right\}$, where $\{y^{1*}, y^{2*}\}$ is the equilibrium allocation that follows. By definition of Γ_θ,

$$\overline{y}^2(\widehat{\theta}^2, y^1) = y_m^2(\mu^{2*}(\widehat{\theta}^2, y^1)) \tag{52}$$

In particular, (52) holds for $y^1 = y^1_m(\widehat{\mu}^1)$. Hence, we can write:

$$\begin{aligned} &E_{\theta^1_{-i},\theta^2}[u^1_i(y^1_m(\widehat{\mu}^1),\theta^1_i) + u^2_i(y^1_m(\widehat{\mu}^1), y^2_m(\mu^{2*}(\widehat{\theta}^2, y^1_m(\widehat{\mu}^1))),\theta^2_i) \mid \theta^1_i] = \\ &E_{\theta^1_{-i},\theta^2}[u^1_i(y^1_m(\widehat{\mu}^1),\theta^1_i) + u^2_i(y^1_m(\widehat{\mu}^1), \overline{y}^2(\widehat{\theta}^2, y^1_m(\widehat{\mu}^1)),\theta^2_i) \mid \theta^1_i] \end{aligned} \tag{53}$$

This implies that any announcement, $\widehat{\theta}^1_1$, to the first mechanism, Γ_θ, when the sequential mechanism is $\{\Gamma^1_\mu, \Gamma^2_\mu\}$, results in the same expected utility as when the sequential mechanism is $\{\Gamma^1_\mu, \Gamma^2_\theta\}$. Therefore, if $\{y^{1*}, y^{2*}\}$ is an equilibrium allocation resulting from $\{\Gamma^1_\mu, \Gamma^2_\mu\}$, it is also an equilibrium allocation when the sequential mechanism is $\{\Gamma^1_\mu, \Gamma^2_\theta\}$.

- **STEP 3: $\{\Gamma^1_\mu, \Gamma^2_\theta\}$ can be replaced by $\{\Gamma^1_\theta, \Gamma^2_\theta\}$.**

Define $\overline{y}(\widehat{\theta}) = y_m(\mu^*(\widehat{\theta}))$, then

$$E_{\theta^1_{-i},\theta^2}[u_i(\overline{y}(\theta),\theta^1_i,\theta^2_i) \mid \theta^1_i] = E_{\theta^1_{-i},\theta^2}[u_i(y_m(\mu^*(\theta)),\theta^1_i,\theta^2_i) \mid \theta^1_i] \tag{54}$$

By expanding u_i, and using

$$\{y^1_m(\mu^{1*}(\theta^1)), y^2_m(\mu^{2*}(\theta^2), y^1_m(\mu^{1*}(\theta^1)))\} = \{y^1_m(\mu^{1*}(\theta^1)), \overline{y}^2(\theta^2, y^1_m(\mu^{1*}(\theta^1)))\} \tag{55}$$

which follows from (51), the right hand side of (54) can be written as

$$E_{\theta^1_{-i},\theta^2}[u^1_i(y^1_m(\mu^{1*}(\theta^1)),\theta^1_i) + u^2_i(y^1_m(\mu^{1*}(\theta^1)), \overline{y}^2(\theta^2, y^1_m(\mu^{1*}(\theta^1))),\theta^2_i) \mid \theta^1_i] =$$

$$\sup_{\mu^1_i \in M^1_i} E_{\theta^1_{-i},\theta^2}[u^1_i(y^1_m(\mu^{1*}_1(\theta^1_1),..,\mu^1_i,..,\mu^{1*}_n(\theta^1_n)),\theta^1_i)$$

$$+u^2_i(y^1_m(\mu^{1*}_1(\theta^1_1),..,\mu^1_i,..,\mu^{1*}_n(\theta^1_n)), \overline{y}^2(\theta^2, y^1_m(\mu^{1*}_1(\theta^1_1),..,\mu^1_i,..,\mu^{1*}_n(\theta^1_n))),\theta^2_i) \mid \theta^1_i] \geq$$

$$\sup_{\theta_i^1 \in \Theta_i^1} E_{\theta_{-i}^1, \theta^2}[u_i^1(\overline{y}^1(\theta_1^1, .., \widehat{\theta}_i^1, .., \theta_n^1), \theta_i^1) +$$

$$u_i^2(\overline{y}^1(\theta_1^1, .., \widehat{\theta}_i^1, .., \theta_n^1), \overline{y}^2(\theta^2, \overline{y}^1(\theta_1^1, .., \widehat{\theta}_i^1, .., \theta_n^1)), \theta_i^2) \mid \theta_1^1]$$

where the equality follows from the definition of a Bayesian Nash equilibrium, and the inequality follows from the fact that the message spaces of Γ_θ^1 are subsets of the message spaces of Γ_μ^1. This completes the proof.